Problems on Algebra III

Winter 2021

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Problem Set 8

Due: Tuesday, January 4, 2022, 2pm

Exercise 1 (Reduced schemes; 3+10 points).
a) Let $(X, \mathscr{O}_X)$ be a scheme. Apply the construction of the induced reduced scheme structure to X itself in order to construct a sheaf $\mathscr{O}_{X^{\mathrm{red}}}$ on X. Discuss the properties of the scheme $(X, \mathscr{O}_{X^{\mathrm{red}}})$.
b) Let $(X, \mathscr{O}_X)$ be a noetherian scheme, and $(X, \mathscr{O}_{X^{\mathrm{red}}})$ the corresponding reduced scheme. Show that $(X, \mathscr{O}_X)$ is affine if and only if $(X, \mathscr{O}_{X^{\mathrm{red}}})$ is affine.

Exercise 2 (Irreducible components; 10 points).
Let $(X, \mathscr{O}_X)$ be a noetherian scheme. The topological space X then decomposes into irreducible components $X_1, ..., X_\ell$. Each one is endowed with the reduced induced scheme structure.
Assume, in addition, that $(X, \mathscr{O}_X)$ is reduced and show that it is affine if and only if its irreducible components $(X_1, \mathscr{O}_{X_1})$, ..., $(X_\ell, \mathscr{O}_{X_\ell})$ are affine.

Exercise 3 (Affine morphisms; 3+2+4+4+4 points).
a) A morphism $(f, f^\#)\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ is *affine*, if there is a covering $(V_i)_{i\in I}$ of Y by affine open subsets, such that $f^{-1}(V_i)$ is affine, $i \in I$. Show that $f^{-1}(V)$ is affine, for every open affine subset $V \subset Y$.
Remark. You will need the elementary characterization of affine schemes, but not the theorem of Serre.
b) Show that a finite morphism (Problem Sheet 5, Exercise 3, b) is affine.
c) Let $(Y, \mathscr{O}_Y)$ be a scheme and $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_Y$-algebras, i.e., a sheaf of rings which is at the same time an $\mathscr{O}_Y$-module. Construct a scheme $(X, \mathscr{O}_X)$ together with a morphism $(f, f^\#)\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$, such that, for every affine open subset $V \subset Y$, there is an isomorphim $\varphi_V\colon f^{-1}(V) \longrightarrow \mathrm{Spec}(\mathscr{A}(V))$, such that, for open affine subsets $V_1 \subset V_2$, the inclusion $f^{-1}(V_1) \subset f^{-1}(V_2)$ corresponds to the restriction homomorphism $\mathscr{A}(V_2) \longrightarrow \mathscr{A}(V_1)$. This scheme is denoted by $\underline{\mathrm{Spec}}(\mathscr{A})$.
d) Let $(f, f^\#)\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ be an affine morphism. Prove that $(X, \mathscr{O}_X)$ is, as a scheme over Y, isomorphic to $\underline{\mathrm{Spec}}(f_\star(\mathscr{O}_X))$.
e) Formulate and prove a universal property of $\underline{\mathrm{Spec}}(\mathscr{A})$.