Problems on Algebra III

Winter 2021

M. Benyoussef, A. Schmitt

Problem Set 7

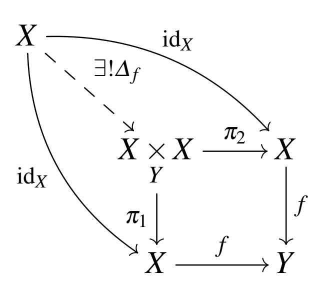
Due: Tuesday, December 14, 2021, 2pm

Exercise 1 (Extension of sections of quasi-coherent sheaves; 10 points).
Let $(X, \mathscr{O}_X)$ be a scheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, $f \in \Gamma(X, \mathscr{O}_X)$, and $s \in \Gamma(X_f, \mathscr{F})$. Suppose that X admits a covering by finitely many affine open schemes $U_1, \ldots, U_\ell$, such that $U_i \cap U_j$ is quasi-compact, $i, j = 1, \ldots, \ell$. Prove that there exist a power $n > 0$ and a section $t \in \Gamma(X, \mathscr{F})$, such that

$$t_{|X_f} = f^n \cdot s.$$

Exercise 2 (Pull back and push forward of quasi-coherent and coherent sheaves; 4+4+4 points).
Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ be schemes and $(f, f^\#) \colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ a morphism.
a) Suppose that $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module. Prove that $f^\star(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module.
b) Suppose that $\mathscr{G}$ is a coherent $\mathscr{O}_Y$-module. Prove that $f^\star(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module.
c) Assume that X is noetherian and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Show that $f_\star(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module. [1]

Exercise 3 (Separated schemes; 6+6+6 points).
Let $(f, f^\#) \colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_X)$ be a morphism of schemes. According to Exercise 1 on Problem Sheet 5, there is a unique morphism $\Delta_f \colon X \longrightarrow X \underset{Y}{\times} X$, called *diagonal*, such that the diagram



commutes. The morphism $(f, f^\#)$ is *separated*, if Δ_f is a closed immersion. A scheme $(X, \mathscr{O}_X)$ is *separated*, if its canonical morphism to $\mathrm{Spec}(\mathbb{Z})$ is separated. According to Hartshorne, Chapter II, Corollary 4.3, this is equivalent to the fact that the image of the diagonal is closed. [2]

---

[1] At the end of last lecture, there was an imprecision in the final argument. It has already been rectified on the slides.

[2] Since the topology on the product $X \times X := X \underset{\mathrm{Spec}(\mathbb{Z})}{\times} X$ is not the product topology and this is, in general, not even the set theoretic product, this is not the classical Hausdorff axiom of topology.

a) Give an example of a non-separated morphism.
b) Let $(X, \mathscr{O}_X)$ be a separated scheme. Prove that the intersection of two affine open subschemes is again an affine open subscheme. Show by means of an example that this is not necessarily true on a non-separated scheme.
c) Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ be schemes and $(f, f^\#), (g, g^\#)\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_X)$ morphisms. Assume that X is reduced and that Y is separated. Prove that the two morphisms are equal if and only if they agree on a dense open subset of X. Find examples to show that this is not true, if X is non-reduced or Y is non-separated.