Problems on Algebra III

Winter 2021

M. Benyoussef, A. Schmitt

Problem Set 6

Due: Tuesday, December 6, 2021, 2pm

Exercise 1 (Cohomology with supports; 2+4+4+4+4+4 points).
Let X be a topological space, $Z \subset X$ a closed subset, and $\mathscr{F}$ a sheaf of abelian groups on X. Set

$$\Gamma_Z(X, \mathscr{F}) := \{ s \in \Gamma(X, \mathscr{F}) \,|\, \mathrm{Supp}(s) \subset Z \}.$$

a) Show that $\Gamma_Z(X, \cdot)$ is a left exact functor from the category $\underline{\mathrm{Sh}}_X$ of sheaves of abelian groups on X to the category $\underline{\mathrm{Ab}}$ of abelian groups. Denote by $H^i_Z(X, \cdot)$ the i-th right derived functor of $\Gamma_Z(X, \cdot)$, $i > 0$.
b) Let

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves of abelian groups on X. Assume that $\mathscr{F}'$ is flasque and prove that the sequence

$$0 \longrightarrow \Gamma_Z(X, \mathscr{F}') \longrightarrow \Gamma_Z(X, \mathscr{F}) \longrightarrow \Gamma_Z(X, \mathscr{F}'') \longrightarrow 0$$

of abelian groups is exact.
c) Prove that $H^i_Z(X, \mathscr{F}) = 0$, for every flasque sheaf $\mathscr{F}$ of abelian groups on X and every $i > 0$.
d) Suppose that $\mathscr{F}$ is a flasque sheaf of abelian groups on X. Show that the sequence

$$0 \longrightarrow \Gamma_Z(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X \setminus Z, \mathscr{F}) \longrightarrow 0$$

of abelian groups is exact.
e) Let $\mathscr{F}$ be a sheaf of abelian groups on X. Prove that there is a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \Gamma_Z(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X \setminus Z, \mathscr{F}) \longrightarrow \\ \longrightarrow H^1_Z(X, \mathscr{F}) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow H^1(X \setminus Z, \mathscr{F}_{|X \setminus Z}) \longrightarrow \\ \longrightarrow H^2_Z(X, \mathscr{F}) \longrightarrow \cdots \end{array} .$$

f) Let $U \subset X$ be an open subset with $Z \subset U$. Prove that, for any $i \in I$, there are functorial isomorphisms

$$H^i_Z(X, \mathscr{F}) \cong H^i_Z(V, \mathscr{F}_{|U}), \quad \mathscr{F} \text{ a sheaf of abelian groups on } X, \quad i \in \mathbb{N}.$$

Exercise 2 (Extension of coherent sheaves; 3+3+4+4+4 points).

Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups on X, and $(\mathscr{F}_i)_{i\in I}$ a family of subsheaves of $\mathscr{F}$. Then, $\mathscr{F}$ is the *union* of the $\mathscr{F}_i$, $i \in I$,

$$\mathscr{F} = \bigcup_{i\in I} \mathscr{F}_i,$$

if

$$\Gamma(U,\mathscr{F}) = \bigcup_{i\in I} \Gamma(U,\mathscr{F}_i)$$

holds, for any open subset $U \subset X$.

a) Let $(X,\mathscr{O}_X)$ be the spectrum of a noetherian ring and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Show that $\mathscr{F}$ is the union of its coherent submodules.

b) Let $(X,\mathscr{O}_X)$ be the spectrum of a noetherian ring, $U \subset X$ an open subset, and $\mathscr{F}$ a coherent $\mathscr{O}_{X|U}$-module. Show that there exists a coherent $\mathscr{O}_X$-module $\mathscr{G}$ with $\mathscr{G}_{|U} \cong \mathscr{F}$.

Hint. Let $\iota\colon U \longrightarrow X$ be the inclusion. Recall that $\iota_\star(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_X$-module.

c) Let $(X,\mathscr{O}_X)$ be a noetherian scheme (compare Exercise 2 from the last problem set), $U \subset X$ an open subset, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_{X|U}$-module, and $\iota\colon U \longrightarrow X$ the inclusion. Show that $\iota_\star(\mathscr{F})$ is quasi-coherent.

d) Let $(X,\mathscr{O}_X)$ be a noetherian scheme, $U \subset X$ an open subset, and $\mathscr{F}$ a coherent $\mathscr{O}_{X|U}$-module. Show that there exists a coherent $\mathscr{O}_X$-module $\mathscr{G}$ with $\mathscr{G}_{|U} \cong \mathscr{F}$.

e) Let $(X,\mathscr{O}_X)$ be a noetherian scheme, $U \subset X$ an open subset, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_{X|U}$-module. Show that $\mathscr{F}$ is the union of its coherent submodules.