# Problems on Algebra III

Winter 2021

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## Problem Set 4

Due: Tuesday, November 23, 2021, 2pm

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Exercise 1 (Gluing sheaves, ringed spaces, and schemes; 5+5+5 points).
a) Let $X$ be a topological space, $(U_i)_{i\in I}$ an open covering of $X$, and $\mathscr{F}_i$ a sheaf of abelian groups on $U_i$, $i \in I$. Suppose we are given isomorphisms

$$\vartheta_{ij}\colon \mathscr{F}_{i|U_i\cap U_j} \longrightarrow \mathscr{F}_{j|U_i\cap U_j}, \quad i,j\in I,$$

such that

$$\vartheta_{ik} = \vartheta_{jk}\circ\vartheta_{ij}$$

holds on $U_i\cap U_j\cap U_k$, $i,j,k\in I$. Construct a sheaf $\mathscr{F}$ on $X$ together with isomorphisms $\eta_i\colon \mathscr{F}_{|U_i} \longrightarrow \mathscr{F}_i$, such that

$$\forall i,j\in I: \quad \vartheta_{ij} = \eta_j\circ\eta_i^{-1} \quad \text{on} \quad U_i\cap U_j.$$

b) Next, let $(X_i, \mathscr{O}_{X_i})_{i\in I}$ be a family of ringed spaces. Suppose that there are open subsets $U_{ij}\subset X_i$ and isomorphisms

$$\vartheta_{ij}\colon (U_{ij}, \mathscr{O}_{X_i|U_{ij}}) \longrightarrow (U_{ji}, \mathscr{O}_{X_j|U_{ji}})$$

of ringed spaces, $i,j\in I$, such that

$$\vartheta_{ik} = \vartheta_{jk}\circ\vartheta_{ij}$$

holds on $\vartheta_{ij}^{-1}(U_{ji}\cap U_{jk})$, $i,j,k\in I$. Show that there are a ringed space $(X,\mathscr{O}_X)$, an open covering $(U_i)_{i\in I}$ of $X$, and isomorphisms $\eta_i\colon (U_i, \mathscr{O}_{X|U_i}) \longrightarrow (X_i, \mathscr{O}_{X_i})$, such that

$$\forall i,j\in I: \quad \vartheta_{ij} = \eta_j\circ\eta_i^{-1} \quad \text{on} \quad U_i\cap U_j.$$

**Hint.** Construct the topological space $X$ first and then apply a).
c) Suppose, in the setting of Part b), that the $(X_i, \mathscr{O}_{X_i})_{i\in I}$ are schemes and that the $\vartheta_{ij}$, $i,j\in I$, are isomorphisms of schemes. Check that the resulting ringed space $(X,\mathscr{O}_X)$ is a scheme.

Exercise 2 (Sheaves of modules and discrete valuation rings; 5+5 points).
Let $R$ be a discrete valuation ring, $K$ is quotient field, and $(X,\mathscr{O}_X)$ its spectrum (as a locally ringed space).
a) Show that giving an $\mathscr{O}_X$-module amounts to specifying an $R$-module $M$, a $K$-vector space $L$, and a homomorphism $\rho\colon M\underset{R}{\otimes} K \longrightarrow L$.

b) Show that the $\mathscr{O}_X$-module defined by $(M,L,\rho)$ is isomorphic to $\widetilde{M}$ if and only if $\rho$ is an isomorphism.

Exercise 3 (Adjoint functors; 7 points).
Let $R$ be a ring and $(X, \mathscr{O}_X)$ its spectrum. Show that the sheafification functor

$$\widetilde{\cdot}\colon \underline{\mathrm{Mod}}_R \longrightarrow \underline{\mathrm{Mod}}_{\mathscr{O}_X}$$

is left adjoint to the global sections functor

$$\Gamma\colon \underline{\mathrm{Mod}}_{\mathscr{O}_X} \longrightarrow \underline{\mathrm{Mod}}_R,$$

i.e., for every $R$-module $M$ and every $\mathscr{O}_X$-module $\mathscr{F}$

$$\mathrm{Hom}_R\big(M, \Gamma(X, \mathscr{F})\big) \cong \mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}).$$

Exercise 4 (Support; 4+4 points).
Let $R$ be a ring, $(X, \mathscr{O}_X)$ its spectrum, $M$ an $R$-module, and $\mathscr{F} = \widetilde{M}$ its sheafification.
a) Let $m \in M$ and $s \in \Gamma(X, \mathscr{F})$ the corresponding global section. Prove that

$$\mathrm{Supp}(s) = V\big(\mathrm{Ann}(m)\big), \quad \mathrm{Ann}(m) := \big\{\, r \in R \,|\, r \cdot m = 0 \,\big\}.$$

b) Assume that $R$ is a noetherian ring and that $M$ is a finitely generated $R$-module. Prove that

$$\mathrm{Supp}(\mathscr{F}) := \big\{\, x \in X \,|\, \mathscr{F}_x \neq 0 \,\big\} = V\big(\mathrm{Ann}(M)\big), \quad \mathrm{Ann}(M) := \big\{\, r \in R \,|\, \forall m \in M:\ r \cdot m = 0 \,\big\}.$$