Problems on Algebra III

Winter 2021

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Problem Set 3

Due: Tuesday, November 16, 2021, 2pm

Exercise 1 (Spectra of discrete valuation rings; 3+3+4 points).
Let k be a field, $R := k[t]$, $S := k[t]_{\langle t\rangle}$ the localization of R at the prime ideal $\langle t\rangle$, and $T := k(t)$ the quotient field of R.
a) Describe the topological space $\mathrm{Spec}(R)$ and the continuous map $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$, coming from the inclusion homomorphism $R \subset S$.
b) Determine the topological space $\mathrm{Spec}(T)$ and the continuous map $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(S)$, coming from the inclusion homomorphism $S \subset T$.
c) Set $X := \mathrm{Spec}(S)$ and $Y := \mathrm{Spec}(T)$. Show that there is a morphism $(Y, \mathscr{O}_Y) \longrightarrow (X, \mathscr{O}_X)$ of ringed spaces which maps the point of Y to the closed point of X and which is not a morphism of locally ringed spaces.

Exercise 2 (Tangent spaces; 10 points).
Let R be a ring, $x \in \mathrm{Spec}(R)$ a point, R_x the local ring at x, $\mathfrak{m}_x \subset R_x$ the maximal ideal, and $k(x) := R_x/\mathfrak{m}_x$ the residue field at x. The *Zariski-tangent space of* $\mathrm{Spec}(R)$ *at* x is

$$T_x\big(\mathrm{Spec}(R)\big) := \mathrm{Hom}_{k(x)}\big(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)\big).$$

In addition, let k be a field and assume that R is a k-algebra. (So, we have a morphism $\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(k)$ of locally ringed spaces.) Finally, let $k[\varepsilon] := k[t]/\langle t^2\rangle$, $\varepsilon := \bar{t}$, be the *algebra of dual numbers over* k.
Show that giving a morphism $\mathrm{Spec}(k[\varepsilon]) \longrightarrow \mathrm{Spec}(R)$ is equivalent to giving a point $x \in \mathrm{Spec}(R)$ with $k(x) = k$, a so-called *k-rational point*, and a tangent vector $v \in T_x(\mathrm{Spec}(R))$.

Exercise 3 (The category of affine schemes; 10 points).
Does the category of affine schemes, i.e., the category of locally ringed spaces which are isomorphic to the spectrum of a ring as locally ringed spaces with morphisms of locally ringed spaces, have an initial and/or terminal object?

Exercise 4 (Fibers of a morphism; 6+4 points).
Let $\varphi\colon R \longrightarrow S$ be a ring homomorphism and $f\colon \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ the associated morphism of locally ringed spaces.
a) How would you define the fiber of f at a closed point $x \in \mathrm{Spec}(R)$ as an affine scheme?
b) Describe the fibers in the case $\varphi\colon \mathbb{C}[t] \longrightarrow \mathbb{C}[t]$, $t \longmapsto t^2$.