# Problems on Algebra III

Winter 2021

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## Problem Set 13

Due: Tuesday, February 8, 2022, 2pm

**Final series!**

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Exercise 1 (Group actions; 5 points).
Let $k$ be a field, $G$ an affine $k$-group scheme, $X$ an affine $k$-scheme, and $\alpha\colon G \underset{k}{\times} X \longrightarrow X$ a morphism. Characterize the property that $\alpha$ be an action by the commutativity of suitable diagrams in $\underline{\mathrm{Aff}}_k$.

Exercise 2 (Actions and representations; 5 points).
Let $k$ be a field, $G$ an affine $k$-group scheme, $X = \mathrm{Spec}(B)$ an affine $k$-scheme, and $\alpha\colon G \underset{k}{\times} X \longrightarrow X$ an action of $G$ on $X$. Make precise the construction of the associated linear action

$$\eta\colon G \times B \longrightarrow B$$

that was sketched in the lecture.

Exercise 3 (Representations of $\alpha_p$; 5 points).
Let $p$ be a prime number, $k$ a field of characteristic $p$, $\alpha_p := \mathrm{Spec}(k[t]/\langle t^p \rangle)$, and $V$ a $k$-vector space. Show that giving a linear action of $\alpha_p$ on $V$ is equivalent to specifying an endomorphism $f\colon V \longrightarrow V$ with $f^p = 0$.

Exercise 4 (Categorical quotients, invariant rings, and $\mathsf{PSL}_2$; 4+8+5+8 points).
Let $k$ be an algebraically closed field, $G = \mathrm{Spec}(A)$ a reduced algebraic[1] $k$-group scheme, $X = \mathrm{Spec}(B)$ a reduced algebraic[2] $k$-scheme, and $\alpha\colon G \underset{k}{\times} X \longrightarrow X$ an action of $G$ on $X$. Set[3]

$$B^G := \big\{ f \in B \,|\, \forall x \in X(k),\ \forall g \in G(k):\ f(g \cdot x) = f(g) \big\}.$$

a) Assume that the categorical quotient[4] $(Z, p)$ for the action of $G$ on $X$ exists. Show that

$$\Gamma(Z, \mathscr{O}_Z) \cong B^G$$

via the homomorphism

$$p^{\#}(Z)\colon \Gamma(Z, \mathscr{O}_Z) \longrightarrow B.$$

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[1] i.e., $A$ is a finitely generated $k$-algebra
[2] i.e., $B$ is a finitely generated $k$-algebra
[3] Compare Exercise 2.
[4] See Problem Set 11, Exercise 2.

b) Assume that $B^G$ is a finitely generated $k$-algebra. Prove that $Z := \mathrm{Spec}(B^G)$ together with the morphism $p\colon X \longrightarrow Z$ induced by the inclusion $B^G \subset B$ is a categorical quotient for $X$ by the action $\alpha$.
c) Show that the categorical quotient $Z$ of $\mathrm{SL}_2(k)$ by the action of the subgroup $\langle \pm \mathbb{E}_2 \rangle$ exists.
d) Assume $1+1 \neq 0$ in $k$. Show that the categorical quotient $Z$ from Part c) has a $k(t)$-rational point which is not represented by a matrix in $\mathrm{SL}_2(k(t))$. Conclude that the group functor

$$\underline{\mathrm{PSL}_2}\colon R \longmapsto \mathrm{SL}_2(R)/\langle \pm \mathbb{E}_2 \rangle$$

is not representable.