Problems on Algebra III

Winter 2021

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Problem Set 12

Due: Tuesday, February 1, 2022, 2pm

Exercise 1 (Base change for group schemes; 6 points).
Let k be a field, G an affine k-group scheme, and $k \longrightarrow K$ a field extension. Show that

$$G \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(K)$$

is an affine K-group scheme.

Exercise 2 (The circle group; 3+3+3+3+3+3 points).
Let k be a field.
a) Show that the functor

$$\begin{array}{rcl} \underline{G}\colon \underline{\mathrm{Alg}_k} & \longrightarrow & \underline{\mathrm{Grps}} \\ R & \longmapsto & \left\{ A \in \mathrm{Mat}(2;R) \,|\, A \cdot A^t = \mathbb{E}_2 \right\} \end{array}$$

is an affine k-group scheme.
b) Show that the determinant yields a homomorphism of G onto μ_2.
c) Verify that

$$\begin{array}{rcl} H\colon \underline{\mathrm{Alg}_k} & \longrightarrow & \underline{\mathrm{Grps}} \\ R & \longmapsto & \left\{ (a,b) \in R \times R \,|\, a^2 + b^2 = 1 \right\} \end{array}$$

with the multiplication

$$(a,b) \cdot (a',b') := (a \cdot a' - b \cdot b', a \cdot b' + a' \cdot b)$$

is an affine k-group scheme which is isomorphic to the kernel of the homomorphism in Part b).
d) Suppose that k contains an element i with $i^2 = -1$. Show that

$$(a,b) \longmapsto a + i \cdot b$$

defines a homomorphism $H \longrightarrow \mathbb{G}_m(k)$. Check that this is an isomorphism, if $\mathrm{Char}(k) \neq 2$.[1]
e) Suppose $\mathrm{Char}(k) = 2$. Show that

$$(a,b) \longmapsto a + b$$

gives a homomorphism $H \longrightarrow \mu_2$ whose kernel is isomorphic to $\mathbb{G}_a(k)$.
f) Assume that H is isomorphic to $\mathbb{G}_m(k)$. Prove that $\mathrm{Char}(k) \neq 2$ and that there is an element $i \in k$ with $i^2 = -1$.

[1] Think, in particular, of $k = \mathbb{C}$.

Exercise 3 (Homomorphisms of Hopf algebras; 7 points).
Let k be a field and $(A, \varepsilon_A, \Delta_A, \sigma_A)$ and $(B, \varepsilon_B, \Delta_B, \sigma_B)$ be Hopf algebras. A k-linear map $\varphi\colon A \longrightarrow B$ is a *homomorphism*, if the diagram

$$\begin{CD} A @>{\Delta_A}>> A \otimes A \\ @V{\varphi}VV @VV{\varphi\otimes\varphi}V \\ B @>{\Delta_B}>> B \otimes B \end{CD}$$

commutes.
Prove that the diagrams

$$\begin{CD} A @>{\varepsilon_A}>> k \\ @V{\varphi}VV @| \\ B @>{\varepsilon_B}>> k \end{CD} \qquad \text{and} \qquad \begin{CD} A @>{\sigma_A}>> A \\ @V{\varphi}VV @VV{\varphi}V \\ B @>{\sigma_B}>> B \end{CD}$$

commute, too.

Exercise 4 (Hopf ideals; 9 points).
Let k be a field and $G = \mathrm{Spec}(A)$ an affine k-group scheme. Characterize the properties that an ideal A must have, so that $\mathrm{Spec}(A/I)$ is a closed subgroup of G. Does $\mathrm{Ker}(\varepsilon\colon A \longrightarrow k)$ have this property? If so, what is the closed subgroup that it defines?