Problems on Algebra II

Winter 2021

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Problem Set 1

Due: Tuesday, October 26, 2021, 2pm

Exercise 1 (Refinements and cohomology; 10 points).
Let X be a topological space, $\mathfrak{U}$, $\mathfrak{V}$ open coverings of X, $\mu, \nu\colon \mathfrak{V} \longrightarrow \mathfrak{U}$ refinements maps, and $\mathscr{F}$ a sheaf of abelian groups on X. Let

$$\mu_n^\star, \nu_n^\star\colon \check{H}^n(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^n(\mathfrak{W}, \mathscr{F})$$

be the induced homomorphisms on Čech cohomology, $n \in \mathbb{N}$.
Prove that

$$\forall n \in \mathbb{N}: \quad \mu_n^\star = \nu_n^\star.$$

Exercise 2 (Vanishing of Čech cohomology; 10 points).
Let X be a topological space, $\mathfrak{U}$ an open covering of X, and $\mathscr{F}$ a sheaf of abelian groups on X. Assume that there is a natural number $n \in \mathbb{N}$, such that the intersection of any $n+2$ distinct open subsets from the covering $\mathfrak{U}$ is empty.
Deduce that

$$\forall k > n \in \mathbb{N}: \quad \check{H}^k(\mathfrak{U}, \mathscr{F}) = 0.$$

Exercise 3 (Čech cohomology of the unit interval and manifolds; 10 points + 15 bonus points).
a) Let $I = [0,1]$ be the unit interval. Prove that any open covering $\mathfrak{U}$ of I admits a refinement $\mathfrak{V}$, such that any three distinct open subsets from the open covering $\mathfrak{V}$ intersect in the empty set.
Infer that $\check{H}^n(I, \mathscr{F}) = 0$, for any sheaf $\mathscr{F}$ of abelian groups on I and any $n > 1$.
b)* Investigate generalizations of the previous result to n-dimensional cubes and manifolds.

Exercise 4 (Constant sheaves on the unit interval; 10 points).
Let $\underline{A}_I$ be the constant sheaf on the unit interval I that is associated with the abelian group A. Prove that $\check{H}^1(I, \underline{A}_I) = 0$.