Problems on Algebra II

Summer 2021

A. Schmitt

Problem Set 9

Due: Monday, June 21, 2021, 4pm

Exercise 1 (Minimal free resolutions; 7+8 points).
Let $(R,\mathfrak{m})$ be a local ring, $k := R/\mathfrak{m}$, and M a finitely generated R-module. A *minimal free resolution* of M is an exact sequence

$$0 \longrightarrow R^{\oplus r_\ell} \xrightarrow{\pi_\ell} R^{\oplus r_{\ell-1}} \xrightarrow{\pi_{\ell-1}} \cdots \xrightarrow{\pi_2} R^{\oplus r_1} \xrightarrow{\pi_1} R^{\oplus r_0} \xrightarrow{\pi_0} M \longrightarrow 0$$

with

$$r_0 := \dim_k(M/\mathfrak{m}\cdot M), \quad r_i := \dim_k(S_i/\mathfrak{m}\cdot S_i), \quad S_i := \ker(\pi_{i-1}), \quad i = 1,\dots,\ell.$$

a) Show that a resolution as above is minimal if and only if

- $\ker(\pi_0) \subset \mathfrak{m}^{\oplus r_0}$,
- $\mathrm{Im}(\pi_{i+1}) = \ker(\pi_i) \subset \mathfrak{m}^{\oplus r_i}$, $i = 0,\dots,\ell-1$.

b) Suppose that the above resolution is minimal. Prove that ℓ agrees with the projective dimension of M.

Exercise 2 (Hilbert's syzygy theorem for polynomial rings; 10 points).
Let k be a field, $n \geq 1$, and $R := k[x_1,\dots,x_n]$ the polynomial ring in n variables.
Quillen–Suslin theorem.[1] *A finitely generated projective module over R is free.*
Prove with the help of the Quillen–Suslin theorem that, for any finitely generated R-module M, there exist an $\ell \leq n$, positive natural numbers $r_1,\dots,r_\ell$, and an exact sequence

$$0 \longrightarrow R^{\oplus r_\ell} \xrightarrow{\pi_\ell} R^{\oplus r_{\ell-1}} \xrightarrow{\pi_{\ell-1}} \cdots \xrightarrow{\pi_2} R^{\oplus r_1} \xrightarrow{\pi_1} R^{\oplus r_0} \xrightarrow{\pi_0} M \longrightarrow 0\cdot$$

Exercise 3 (A stably free module which is not free; 6+9 points).
Define $R := \mathbb{R}[x,y,z]/\langle x^2+y^2+z^2-1\rangle$ and $T := \{\, (r,s,t) \in R^3 \,|\, r\cdot x + s\cdot y + t\cdot z = 0\,\}$.
a) Show that $R \oplus T \cong R\cdot(x,y,z) \oplus T = R^{\oplus 3}$.[2]
Hint. The inner product

$$\begin{aligned} \langle\cdot,\cdot\rangle\colon R^{\times 3}\times R^{\times 3} &\longrightarrow R \\ \big((r,s,t),(u,v,w)\big) &\longmapsto r\cdot u + s\cdot v + t\cdot w \end{aligned}$$

[1] Theorem 3.15 in Kunz, Ernst: *Introduction to commutative algebra and algebraic geometry*, translated from the 1980 German original by M. Ackerman, with a preface by D. Mumford, reprint of the 1985 edition, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2013, xiv+238 pp.

[2] It is important that the second relation is equality.

(appearing in the definition of T) might be useful.

Interpretation. The ring R is the ring of real valued polynomial functions on the unit sphere $S^2 = \{\, (a,b,c) \in \mathbb{R}^3 \,|\, a^2+b^2+c^2 = 1 \,\}$, and T corresponds to the tangent bundle of S^2. The normal bundle N to S^2 is trivial, $N \oplus T$ is isomorphic to the restriction of the tangent bundle of $\mathbb{R}^3$ to S^2, and the tangent bundle of $\mathbb{R}^3$ is trivial.

b) Prove that $T \ncong R^2$.

Hint. Assume that $T \cong R^{\oplus 2}$, pick a basis $(f_1,g_1,h_1),(f_2,g_2,h_2)$ of T, and show that the determinant

$$\begin{pmatrix} x & y & z \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{pmatrix}$$

is a unit in R. Infer that

$$\begin{aligned} (f_1,g_1,h_1)\colon S^2 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \big(f_1(x),g_1(x),h_1(x)\big) \end{aligned}$$

is a differentiable map which is everywhere distinct from $(0,0,0)$. This contradicts the **hairy ball theorem** (see, e.g., Alexander Schmitt, Analysis III, 5.9.1 Satz, `http://userpage.fu-berlin.de/~aschmitt/`).