# Problems on Algebra II

Summer 2021

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## Problem Set 8

Due: Monday, June 14, 2021, 4pm

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Exercise 1 (Group cohomology; 3+2+7+3 points).
This exercise continues Exercise 1 from Problem Set 5.
a) Let $G$ be a group and Inv the left exact functor from the category of $G$-modules to the category of abelian groups that assigns to a $G$-module $M$ the group $M^G$ of $G$-invariant elements of $M$. Show that the right derived functors of Inv are given by $\mathrm{Ext}^k_{\mathbb{Z}[G]}(\mathbb{Z},\cdot)$, $k \in \mathbb{N}$.
b) Explain how the right derived functors of Inv may be computed with the help of a projective resolution of $\mathbb{Z}$ endowed with the trivial $G$-module structure.
c) Let $S$ be a non-empty set and $E_k$ the free $\mathbb{Z}$-module with basis $S^{\times(k+1)}$, i.e., the set of $(k+1)$-tuples $(x_0,\dots,x_k)$ with entries in $S$, $k \geq \mathbb{N}$. Define

$$d_{k+1}\colon E_{k+1} \longrightarrow E_k$$

as the homomorphism that is induced by the map

$$(x_0,\dots,x_{k+1}) \longmapsto \sum_{j=0}^{k}(-1)^j\cdot(x_0,\dots,x_{j-1},x_{j+1},\dots,x_{k+1})$$

on the basis vectors, and

$$d_0\colon E_0 \longrightarrow \mathbb{Z}$$

as the homomorphism that is induced by $d_0(x_0)=1$, $x_0 \in S$. Show that the sequence

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact.
**Hint.** First show that you get a complex. For exactness, pick an element $z \in S$ and define

$$h_k\colon E_k \longrightarrow E_{k+1}$$

by

$$(x_0,\dots,x_k) \longmapsto (z,x_0,\dots,x_k)$$

and check that

$$d_{k+1}\circ h_k + h_{k-1}\circ d_k = \mathrm{id}_{E_k}, \quad k \in \mathbb{N}.$$

d) In the set-up of Part a), b), use $S=G$ in Part c) and let $G$ act on $E_{k+1}$ by

$$g\cdot(x_0,\dots,x_k) := (g\cdot x_0,\dots,g\cdot x_k), \quad k \in \mathbb{N}.$$

Verify that $E_k$ is a free $\mathbb{Z}[G]$-module, $k \in \mathbb{N}$, and conclude that the resolution from Part c) is a projective resolution of the trivial $G$-module $\mathbb{Z}$.

Exercise 2 (Cohomology of cyclic groups; 4+6 points).
Let $n \geq 1$ and $C_n := \langle t \mid t^n = 1 \rangle$ the cyclic group of order $n$. Define the elements

$$t-1 \quad \text{and} \quad N := 1 + t + \cdots + t^{n-1}$$

in the group ring $\mathbb{Z}[C_n]$.[1]

a) Set $\varepsilon\colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$, $\sum\limits_{g\in G} a_g \cdot \varepsilon_g \longmapsto \sum\limits_{g\in G} a_g$. Show that

$$\cdots \xrightarrow{(t-1)} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{(t-1)} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective resolution of the trivial $G$-module $\mathbb{Z}$.

b) Use the complex of Part a) to compute the cohomology groups of $C_n$.

Exercise 3 (The global dimension of the ring of integers and applications; 5+2+4+4 points).
a) Determine the global dimension of the ring of integers.

b) Prove that $\operatorname{Tor}_k^{\mathbb{Z}}(A,B) = 0$, for any two abelian groups $A, B$ and any natural number $k \geq 2$.

c) Show that[2]

$$\operatorname{Tor}_1^{\mathbb{Z}}\big(A, \mathbb{Z}/\langle n\rangle\big) = A[n] = \big\{\, a \in A \mid n \cdot a = 0 \,\big\},$$

for every abelian group $A$ and every natural number $n$.

d) Let

$$(\star):\quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups and $n \in \mathbb{N}$. Complete the sequence $(\star) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/\langle n\rangle$ to the long exact Tor-sequence and specify all maps in that sequence.

[1] Observe that the group $C_n$ is written multiplicatively and that addition refers to addition in the group ring.
[2] Compare Exercise 2 from Problem Set 5.