Problems on Algebra II

Summer 2021

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Problem Set 7

Due: Monday, June 7, 2021, 4pm

Exercise 1 (Examples of tensor products; 3+5+4 points).
a) Let n be a natural number. Compute the tensor products

$$\mathbb{Q} \otimes_{\mathbb{Z}} \big(\mathbb{Z}/\langle n\rangle\big) \quad \text{and} \quad (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \big(\mathbb{Z}/\langle n\rangle\big).$$

b) Let m and n be natural numbers. Determine the tensor product

$$\big(\mathbb{Z}/\langle m\rangle\big) \otimes_{\mathbb{Z}} \big(\mathbb{Z}/\langle n\rangle\big).$$

c) Is the tensor product of the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n \longmapsto 2n} \mathbb{Z} \longrightarrow \mathbb{Z}/\langle 2\rangle \longrightarrow 0$$

with $\mathbb{Z}/\langle 2\rangle$ exact?

Exercise 2 (Some properties of Hom and $\otimes$; 4+4+4 points).
Let R be a commutative ring.
a) Let A, B, and C be R-modules. Verify that there is a canonical isomorphism

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R\big(A, \mathrm{Hom}_R(B, C)\big).$$

b) Let A be an R-module and B an abelian group. Show that there is a natural isomorphism

$$\mathrm{Hom}_R\big(A, \mathrm{Hom}_{\mathbb{Z}}(R, B)\big) \cong \mathrm{Hom}_{\mathbb{Z}}(A, B)$$

of R-modules.
c) Let A, B, and C be R-modules. Prove that

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C)$$

(The tensor product is associative.)

Exercise 3 (The tensor product of algebras; 6 points).
Let R be a commutative ring and A and B R-algebras. Show that there is an R-algebra structure on $A \otimes_R B$ with

$$(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb'), \quad \forall a, a' \in A,\ b, b' \in B.$$

Exercise 4 (Tor; 6+4 points).
Let R be a commutative ring.
a) Show that

$$\operatorname{Tor}_n^R(M,-) \cong L_n(- \otimes_R M), \quad n \in \mathbb{N},\ M \text{ an } R\text{-module.}$$

(This means that Tor may be computed by taking projective resolutions in the first variable.)
b) Verify that

$$\operatorname{Tor}_n^R(M,N) \cong \operatorname{Tor}_n^R(N,M)$$

holds for all $n \in \mathbb{N}$ and all R-modules M and N.