Problems on Algebra II

Summer 2021

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Problem Set 6

Due: Monday, May 31, 2021, 4pm

Exercise 1 (Double complexes and Ext; 8+7 points).
Let $\mathscr{A}$ be an abelian category. A *double complex in* $\mathscr{A}$ consists of

- objects K^{ij}, $i, j \in \mathbb{Z}$,
- vertical differentials $d'_{ij}\colon K^{ij} \longrightarrow K^{i+1\,j}$, $i, j \in \mathbb{Z}$,
- horizontal differentials $d''_{ij}\colon K^{ij} \longrightarrow K^{i\,j+1}$, $i, j \in \mathbb{Z}$,

such that

- $d'_{i+1\,j} \circ d'_{ij} = 0$, $i, j \in \mathbb{Z}$,
- $d''_{i\,j+1} \circ d''_{ij} = 0$, $i, j \in \mathbb{Z}$,
- $d''_{i+1\,j} \circ d'_{ij} = d'_{i\,j+1} \circ d''_{ij}$, $i, j \in \mathbb{Z}$.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & {\scriptstyle d'_{i+1\,j-1}}\uparrow & & {\scriptstyle d'_{i+1\,j}}\uparrow & & {\scriptstyle d'_{i+1\,j+1}}\uparrow & & \\
\cdots & \xrightarrow{d''_{i+1\,j-2}} & K^{i+1\,j-1} & \xrightarrow{d''_{i+1\,j-1}} & K^{i+1\,j} & \xrightarrow{d''_{i+1\,j}} & K^{i+1\,j+1} & \xrightarrow{d''_{i+1\,j+1}} & \cdots \\
 & & {\scriptstyle d'_{i\,j-1}}\uparrow & & {\scriptstyle d'_{ij}}\uparrow & & {\scriptstyle d'_{i\,j+1}}\uparrow & & \\
\cdots & \xrightarrow{d''_{i\,j-2}} & K^{i\,j-1} & \xrightarrow{d''_{i\,j-1}} & K^{ij} & \xrightarrow{d''_{ij}} & K^{i\,j+1} & \xrightarrow{d''_{i\,j+1}} & \cdots \\
 & & {\scriptstyle d'_{i-1\,j-1}}\uparrow & & {\scriptstyle d'_{i-1\,j}}\uparrow & & {\scriptstyle d'_{i-1\,j+1}}\uparrow & & \\
\cdots & \xrightarrow{d''_{i-1\,j-2}} & K^{i-1\,j-1} & \xrightarrow{d''_{i-1\,j-1}} & K^{i-1\,j} & \xrightarrow{d''_{i-1\,j}} & K^{i-1\,j+1} & \xrightarrow{d''_{i-1\,j+1}} & \cdots \\
 & & {\scriptstyle d'_{i-2\,j-1}}\uparrow & & {\scriptstyle d'_{i-2\,j}}\uparrow & & {\scriptstyle d'_{i-2\,j+1}}\uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

a) Assume

- $K_{ij} = 0$ for $i < -1$ or $j < -1$,
- the complex $(K^{\bullet j}, d'_{\bullet j})$ is exact for all $j \geq 0$,
- the complex $(K^{i\bullet}, d''_{i\bullet})$ is exact for all $i \geq 0$.

Prove that

$$H^n(K^{\bullet -1}, d'_{\bullet -1}) \cong H^n(K^{-1\bullet}, d''_{-1\bullet}), \quad n \geq 0.$$

b) Use this result to show that

$$\mathrm{Ext}^n_R(M,N) \cong \mathrm{Ext}^n_{\underline{\mathrm{Mod}}_R^{\mathrm{op}}}(N,M)$$

for all n and all R-modules M and N.
(Use an injective resolution of N and a projective resolution of M in order to produce a double complex as in a).)

Exercise 2 (Torsion groups; 4+6 points).
Let A be an abelian group and p a prime number. The *p-primary subgroup* of A is

$$A_p := \left\{ a \in A \mid \exists n \in \mathbb{N} : \ \mathrm{ord}(a) = p^n \right\}.$$

a) Describe the p-primary subgroup of $\mathbb{Q}/\mathbb{Z}$, p a prime number.
b) Let A be an abelian torsion group. Show that

$$A \cong \bigoplus_{p \text{ prime}} A_p.$$

Exercise 3 (Divisible groups; 3+4+4+4 points).
a) Are there finitely generated divisible abelian groups?
b) Show that a torsion free divisible abelian group is a vector space over $\mathbb{Q}$.
c) Prove that a divisible abelian group is isomorphic to the direct sum of a divisible abelian torsion group and $\bigoplus_{i \in I} \mathbb{Q}$ for a suitable index set I.
d) Let D be a divisible abelian torsion group and p a prime number. Show that the p-primary subgroup D_p of D is divisible, as well.