Problems on Algebra II

Summer 2021

A. Schmitt

Problem Set 5

Due: Monday, May 25, 2021, 10am

Exercise 1 (Group cohomology; 5+5+5 points).
Let G be a (not necessarily) abelian group with neutal element e. A *G-module* is an abelian group M together with a map

$$\begin{aligned} \alpha\colon G\times M &\longrightarrow M \\ (g,m) &\longmapsto g\cdot m, \end{aligned}$$

such that

- $\forall g\in G, \forall m,m'\in M: g\cdot(m+m') = g\cdot m + g\cdot m'$,
- $\forall m\in M: e\cdot m = m$,
- $\forall g,g'\in G, \forall m\in M: (g\cdot g')\cdot m = g\cdot(g'\cdot m)$.

Homomorphisms between G-modules are G-equivariant group homomorphisms.
a) The *group algebra* $\mathbb{Z}[G]$ of G is the free abelian group $\bigoplus_{g\in G}\mathbb{Z}\cdot\varepsilon_g$ together with the multiplication that is determined by the rule

$$\forall g,g'\in G:\quad \varepsilon_g\cdot\varepsilon_{g'} := \varepsilon_{g\cdot g'}.$$

Explain how to identify G-modules with right $\mathbb{Z}[G]$-modules. Conclude that the category G-Mod of G-modules with homomorphisms is an abelian category which has enough injectives and projectives.
b) For a G-module M, the set of *G-invariant elements* is

$$M^G := \big\{\, m\in M \,|\, \forall g\in G:\ g\cdot m = m \,\big\}.$$

Show that M^G is a G-submodule of M.
Endow $\mathbb{Z}$ with the trivial G-module structure, i.e., $g\cdot k = k$, $g\in G$, $k\in\mathbb{Z}$. Show that

$$\begin{aligned} \mathrm{Hom}_G(\mathbb{Z},M) &\longrightarrow M^G \\ \varphi &\longmapsto \varphi(1) \end{aligned}$$

is an isomorphism of G-modules. Infer that $M\longmapsto M^G$ is a left exact functor.[1]
c) Find an example of a group G and a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

[1] The right derived functors of this functor are denoted by $H^i(G,\cdot)$, $i\in\mathbb{N}$.

of G-modules, such that

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

is not right exact.
Hint. You may take $G = \mathbb{Z}/(2 \cdot \mathbb{Z})$.

Exercise 2 (Torsion; 5 points).
Fix $n \geq 1$. For an abelian group A, set

$$A[n] := \{ a \in A \,|\, n \cdot a = 0 \}.$$

Check that $A \longmapsto A[n]$ gives rise to a covariant functor from $\underline{\text{Ab}}$ to $\underline{\text{Ab}}$. What are the exactness properties of this functor?

Exercise 3 (Degree shift; 10 points).
Let $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ be a left exact functor between abelian categories, and

$$0 \longrightarrow A \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_r \longrightarrow B \longrightarrow 0$$

an exact sequence in $\mathscr{A}$. Suppose that I_1, ..., I_r are injective objects. Show that

$$R^n F(B) \cong R^{n+r} F(A), \quad n \geq 1.$$

Formulate the corresponding result for projective objects instead of injective objects.

Exercise 4 (Composed functors; 10 points).
Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be abelian categories, $F \colon \mathscr{A} \longrightarrow \mathscr{B}$ a left exact functor, and $G \colon \mathscr{B} \longrightarrow \mathscr{C}$ an exact functor. Assume that $\mathscr{A}$ has enough injectives and prove that

$$\forall i \in \mathbb{N} : \quad R^i(G \circ F) \cong G \circ R^i(F).$$