Problems on Algebra II

Summer 2021

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Problem Set 4

Due: Monday, May 17, 2021, 4pm

Exercise 1 (The snake lemma; 10 points).
Let $\mathscr{A}$ be an abelian category and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ \| & & \downarrow f & & \downarrow g & & \downarrow h & & \| \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

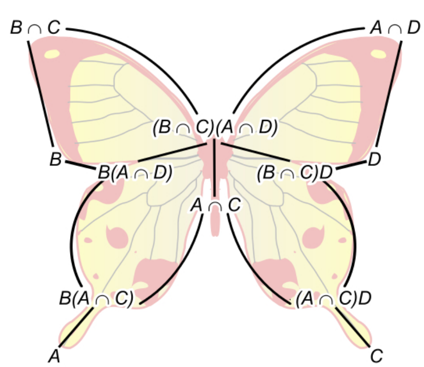
a commutative diagram in $\mathscr{A}$ with exact rows. Show that it induces an exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h) \longrightarrow 0.$$

Exercise 2 (The Zassenhaus lemma; 6+4 points).
Zassenhaus lemma.[1]
Let G be a not-necessarily commutative group, $A, C \subset G$ subgroups, and $B \triangleleft A$, $D \triangleleft C$ ***normal*** *subgroups. Then,*

$$\frac{(A\cap C)\cdot B}{(A\cap D)\cdot B} \cong \frac{(A\cap C)\cdot D}{(B\cap C)\cdot D}.$$

The Zassenhaus lemma is visualized by the diagram[2]



in which downward edges stand for inclusions. (Because of the shape of the diagram, the result is also refered to as butterfly lemma.)

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[1] From: H. Zassenhaus, *Zum Satz von Jordan–Hölder–Schreier*, Abh. Math. Sem. Univ. Hamburg **10** (1934), no. 1, 106-8. (Hans Julius Zassenhaus (1912 - 1991), German mathematician.)

[2] By Claudio Rocchini, `https://commons.wikimedia.org/wiki/File:Butterfly_lemma.svg`.

a) Prove the lemma (for non-commutative groups).
b) How would you formulate the lemma in an abelian catgory?

Exercise 3 (Injectives and projectives; 5+5 points).
Let R be a noetherian ring and $\underline{\text{fgMod}}_R$ the abelian category of **finitely generated** R-modules.
a) Does $\underline{\text{fgMod}}_R$ have enough projectives?
b) Does $\underline{\text{fgMod}}_R$ have enough injectives? (Consider the special cases that R is a field and $R = \mathbb{Z}$.)

Exercise 4 (Projective quiver representations; 10 points).
Determine the projective representations of the quiver $\bullet \longrightarrow \bullet$ in the category $\underline{\text{Vect}}_k$, k a field.