Problems on Algebra II

Summer 2021

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Problem Set 3

Due: Monday, May 10, 2021, 4pm

Exercise 1 (Abelian subcategories; 5+5 points).
a) Is the category of finite abelian groups an abelian subcategory of the category of abelian groups?
b) Let R be a commutative ring with unit. Is the category of finitely generated R-modules always an abelian subcategory of the category of R-modules?
Definitions. Let $\mathscr{C}$ be a category. A *subcategory* $\mathscr{D}$ of $\mathscr{C}$ is specified by a subclass $\mathrm{Ob}(\mathscr{D}) \subset \mathrm{Ob}(\mathscr{C})$ and, for any two objects $X, Y \in \mathrm{Ob}(\mathscr{D})$, a subset $\mathrm{Mor}_{\mathscr{D}}(X,Y) \subset \mathrm{Mor}_{\mathscr{C}}(X,Y)$, such that

- $\forall X \in \mathrm{Ob}(\mathscr{D})$: $\mathrm{id}_X \in \mathrm{Mor}_{\mathscr{D}}(X,X)$,
- $\forall X,Y,Z \in \mathrm{Ob}(\mathscr{D})$, $\forall f \in \mathrm{Mor}_{\mathscr{D}}(X,Y), g \in \mathrm{Mor}_{\mathscr{D}}(Y,Z)$: $g \circ f \in \mathrm{Mor}_{\mathscr{D}}(X,Z)$.

Then, $\mathscr{C}$ is itself a category. We say that $\mathscr{C}$ is a *full subcategory* of $\mathscr{C}$, if

$$\mathrm{Mor}_{\mathscr{D}}(X,Y) = \mathrm{Mor}_{\mathscr{C}}(X,Y),$$

for all $X, Y \in \mathrm{Ob}(\mathscr{D})$.
If $\mathscr{A}$ is an abelian category and $\mathscr{B}$ is a subcategory, we say that $\mathscr{B}$ is an *abelian subcategory*, if

- it contains the null object,
- for all $A, B \in \mathrm{Ob}(\mathscr{B})$, $\mathrm{Mor}_{\mathscr{D}}(X,Y)$ is a subgroup $\mathrm{Mor}_{\mathscr{C}}(X,Y)$,
- for all $A, B \in \mathrm{Ob}(\mathscr{B})$, the direct sum $A \oplus B$ of A and B in $\mathscr{A}$ is contained in $\mathscr{B}$,
- for all $A, B \in \mathrm{Ob}(\mathscr{B})$ and any morphism $f\colon A \longrightarrow B$ in $\mathscr{B}$, the kernel and the cokernel of g in $\mathscr{A}$ are contained in $\mathscr{B}$.

Observe that $\mathscr{B}$ will then be an abelian category.

Exercise 2 (Abelian categories; 7+3 points).
a) Let $\mathscr{D}$ be a small category and $\mathscr{A}$ an abelian category. Show that the category $\mathrm{Fun}(\mathscr{D}, \mathscr{A})$ of covariant functors from $\mathscr{D}$ to $\mathscr{A}$ is again an abelian category.
b) Let $\mathscr{A}$ be an abelian category. Show that the category of complexes in $\mathscr{A}$ is also an abelian category.

Exercise 3 (A complex of abelian groups; 5 points).
Consider the abelian groups

$$C^k := \begin{cases} 0, & \text{if } k < 0 \\ \mathbb{Z}/\langle 8 \rangle, & \text{if } k \geq 0 \end{cases}, \quad k \in \mathbb{Z},$$

and, for $k \geq 0$, the homomorphisms

$$\begin{array}{rcl} \delta^k \colon C^k & \longrightarrow & C^{k+1} \\ [\ell] & \longmapsto & [4 \cdot \ell]. \end{array}$$

For $k < 0$, $\delta^k \colon C^k \longrightarrow C^{k+1}$ is defined in the obvious way. Show that $(C^\bullet, \delta^\bullet)$ is a complex of abelian groups and compute its cohomology groups.

Exercise 4 (Complexes of vector spaces; 5+10 points).
Let K be a field.
a) Suppose we are given K-vector spaces $(B^k)_{k \in \mathbb{Z}}$ and $(H^k)_{k \in \mathbb{Z}}$. For $k \in \mathbb{Z}$, set $C^k := B^k \oplus H^k \oplus B^{k-1}$ and

$$\begin{array}{rcl} \delta^k \colon C^k & \longrightarrow & C^{k+1} \\ (a, b, c) & \longmapsto & (0, 0, a). \end{array}$$

Show that $(C^\bullet, \delta^\bullet)$ is a complex of K-vector spaces with

$$H^k(C^\bullet, \delta^\bullet) \cong H^k, \quad k \in \mathbb{Z}.$$

b) Let $(E^\bullet, \varepsilon^\bullet)$ be a complex of K-vector spaces. Prove that it is isomorphic to the complex from a) that is constructed from $(B^k(E^\bullet, \varepsilon^\bullet))_{k \in \mathbb{Z}}$ and $(H^k(E^\bullet, \varepsilon^\bullet))_{k \in \mathbb{Z}}$.