Problems on Algebraic Geometry

Summer 2021

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Problem Set 2

Due: Monday, May 3, 2021, 4pm

Exercise 1 (Mono- and epimorphisms in Sets; 8 points).
Prove that a map $\alpha\colon A \longrightarrow B$ in the category Sets of sets is a monomorphism if and only if it is injective and an epimorphism if and only if it is surjective.

Exercise 2 (Free abelian groups; 4+6 points).
Let $\mathcal{C}$ be the category whose objects are free abelian groups of finite rank and whose morphisms are group homomorphisms.
a) Is $\mathcal{C}$ an additive category?
b) Give an example of a monomorphism $\alpha\colon A \longrightarrow B$ in $\mathcal{C}$ which is both a mono- and an epimorphism, but not an isomorphism.

Exercise 3 (The null object in an additive category; 7 points).
Let $\mathcal{A}$ be an additive category. Recall that there is an element $0 \in \mathrm{Ob}(\mathcal{A})$ with $\mathrm{Mor}_{\mathcal{A}}(0,0) = \{0\}$.[1] Show that, for every $A \in \mathrm{Ob}(\mathcal{A})$,

$$\mathrm{Mor}_{\mathcal{A}}(0,A) = \{0\} \quad \text{and} \quad \mathrm{Mor}_{\mathcal{A}}(A,0) = \{0\}.$$

This means that 0 is a **null object** in the category $\mathcal{A}$.

Exercise 4 (Direct sums and products; 5+5+5 points).
a) Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Ob}(\mathcal{C})$. A triple (C, ι_1, ι_2), consisting of $C \in \mathrm{Ob}(\mathcal{C})$, $\iota_1\colon A \longrightarrow C$ and $\iota_2\colon B \longrightarrow C$ is said to be the *direct sum of* A *and* B, if for every object D of C and every pair of morphisms $\varphi_1\colon A \longrightarrow D$ and $\varphi_2\colon B \longrightarrow D$, there is a unique morphism $\Phi\colon C \longrightarrow D$ with

$$\varphi_1 = \Phi \circ \iota_1 \quad \text{and} \quad \varphi_2 = \Phi \circ \iota_2.$$

Notation: $A \oplus B := C$.
Define the dual notion of a **direct product** $A \sqcap B$.
b) Let $\mathcal{A}$ be an **additive** category. Show that, for any two objects A and B of $\mathcal{A}$, the direct sum $A \oplus B$ comes with morphisms $\pi_1\colon A \oplus B \longrightarrow A$ and $\pi_2\colon A \oplus B \longrightarrow B$, such that $(A \oplus B, \pi_1, \pi_2)$ is the direct product of A and B.

[1] Of course, $0\colon 0 \longrightarrow 0$ stands for id_0.

c) Let $\mathcal{A}$ be an additive category, $A, B, C \in \mathrm{Ob}(\mathcal{A})$, and $\iota_1 \colon A \longrightarrow C$, $\iota_2 \colon B \longrightarrow C$, $\pi_1 \colon C \longrightarrow A$, and $\pi_2 \colon C \longrightarrow B$ morphisms. Suppose

$$\pi_i \circ \iota_j = \begin{cases} 0, & \text{if } i \neq j \\ \mathrm{id}, & \text{if } i = j \end{cases}, \quad i, j \in \{1, 2\}, \quad \text{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_C$$

Show that (C, ι_1, ι_2) is the direct sum of A and B.