Problems on Algebra II

Summer 2021

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Problem Set 12 (Last problem set)

Due: Monday, July 12, 2021, 4pm

Exercise 1 (Cohomology and push forward; 9 points).
Let X be a topological space, Z a closed subset of X, and $j\colon Z \longrightarrow X$ the inclusion. Show that, for a sheaf $\mathscr{F}$ of abelian groups on Z, one has

$$H^i(Z, \mathscr{F}) \cong H^i(X, j_\star \mathscr{F}), \quad i \geq 0.$$

Exercise 2 (Soft sheaves and fine sheaves; 13 points).
Let X be a **paracompact** topological space. Check that a sheaf $\mathscr{F}$ of abelian groups on X is fine if and only if $\mathscr{H}om_{\underline{\mathrm{Sh}}_X}(\mathscr{F}, \mathscr{F})$ is soft.

Exercise 3 (Vector bundles; 3+3+3 points).
Let X be a differentiable manifold and E, F vector bundles on X.
a) Define the notion of a *homomorphism* from E to F.
b) Let $(U_i, \varphi_i)_{i\in I}$ and $(U_i, \psi_i)_{i\in I}$ be trivializations for E and F, respectively. Describe a homomorphism from E to F in terms of differentiable maps $U_i \longrightarrow \mathrm{Mat}(n, m, \mathbb{R})$, $i \in I$.
c) Do vector bundles on X form an abelian category?

Exercise 4 (The Picard group; 9 points).
Let X be a topological **manifold**. Prove that

$$\mathrm{Pic}(X) \cong H^1(X, \mathscr{C}_X^\star) \cong H^2(X, \underline{\mathbb{Z}}_X).$$