Problems on Algebra II

Summer 2021

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Problem Set 11

Due: Monday, July 5, 2021, 4pm

Exercise 1 (Extension by zero; 10 points).
Let X be a topological space, $j\colon U \subset X$ an open subset, and $i\colon Z \subset X$, $Z := X \setminus U$.
a) Let $\mathscr{F}$ be a sheaf on U. Show that, for every sheaf $\mathscr{G}$ on X, one has

$$\operatorname{Hom}_{\underline{\mathrm{Sh}}_X}(j_!\mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}_{\underline{\mathrm{Sh}}_U}(\mathscr{F}, \mathscr{G}_{|U}).$$

b) Let $\mathscr{F}$ be a sheaf on X. Prove that there is an exact sequence

$$0 \longrightarrow j_!(\mathscr{F}_{|U}) \longrightarrow \mathscr{F} \longrightarrow i_\star(\mathscr{F}_{|Z}) \longrightarrow 0$$

of sheaves on X.

Exercise 2 (Hom; 3+2+2+3 points).
Let X be a topological space and $\mathscr{F}$ and $\mathscr{G}$ sheaves of abelian groups on X.
a) Show that

$$\mathscr{H}om(\mathscr{F}, \mathscr{G}) : U \longmapsto \mathscr{H}om_{\underline{\mathrm{Sh}}_U}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$$

is a **sheaf** of abelian groups on X. (Why doesn't one use $U \longmapsto \operatorname{Hom}_{\underline{\mathrm{Ab}}}(\mathscr{F}(U), \mathscr{G}(U))$?)
b) Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ and $\mathscr{G}$ sheaves of $\mathscr{O}_X$-modules. Verify that

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) : U \longmapsto \mathscr{H}om_{\underline{\mathrm{Mod}}_{\mathscr{O}_{X|U}}}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$$

is an $\mathscr{O}_X$-module.
c) Let $(X, \mathscr{O}_X)$ be a ringed space. Check that

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}$$

holds for every $\mathscr{O}_X$-module $\mathscr{F}$.
d) Let $(f, f^\#)\colon (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. Show that, for an $\mathscr{O}_Y$-module $\mathscr{F}$ and an $\mathscr{O}_X$-module $\mathscr{G}$, one has

$$\operatorname{Hom}_{\mathscr{O}_X}(f^\star\mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{F}, f_\star\mathscr{G}).$$

Exercise 3 (Stalks of the tensor product; 10 points).
Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ and $\mathscr{G}$ two $\mathscr{O}_X$-modules. Prove that, for every point $x \in X$,

$$(\mathscr{F} \underset{\mathscr{O}_X}{\otimes} \mathscr{G})_x \cong \mathscr{F}_x \underset{\mathscr{O}_{X,x}}{\otimes} \mathscr{G}_x.$$

Exercise 4 (The Picard group; 10 points).
Let $(X, \mathscr{O}_X)$ be a ringed space. Set

$$\mathrm{Pic}(X) := \Big\{ \text{Isomorphism classes of invertible sheaves on } X \Big\}.$$

Show that the tensor product induces the structure of an abelian group on $\mathrm{Pic}(X)$ with neutral element $[\mathscr{O}_X]$ and inverse

$$[\mathscr{L}]^{-1} = [\mathscr{L}^\vee], \quad \mathscr{L}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X).$$