Problems on Algebra II

Summer 2021

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Problem Set 10

Due: Monday, June 28, 2021, 4pm

Exercise 1 (Presheaves which are not sheaves; 8 points).
Give an example of a topological space X and a presheaf $\mathscr{F}$ on X which satisfies the second sheaf axiom ("gluing") but not the first one.

Exercise 2 (Sheaves on $[0,1]$; 5+5 points).
a) Let the interval $I := [0,1]$ be endowed with the metric topology coming from the absolute value. Show that there is a unique sheaf of abelian groups $\mathscr{F}$ on I with stalks $\mathscr{F}_0 = \mathbb{Z} = \mathscr{F}_1$ and $\mathscr{F}_x = 0$, for $x \in (0,1)$.
b) Let $\mathscr{G} := \underline{\mathbb{Z}}_I$ be the constant sheaf on I associated with the group $\mathbb{Z}$ and $\mathscr{F}$ the sheaf from Part a). Describe the sheaf homomorphisms from $\mathscr{F}$ to $\mathscr{G}$ and from $\mathscr{G}$ to $\mathscr{F}$.

Exercise 3 (Sheaves on S^1; 5+5 points).
Let $p \in S^1$ be a point, $U := S^1 \setminus \{p\}$, $j\colon U \longrightarrow S^1$ the inclusion, and $\mathscr{S}_p$ the skyscraper sheaf with stalk $\mathbb{Z}$ at p.
a) Show that there is an exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}_X \longrightarrow j_\star \underline{\mathbb{Z}}_U \longrightarrow \mathscr{S}_p \longrightarrow 0.$$

b) Compute the left exact sequence which results from applying $\Gamma(I,-)$ to the above exact sequence, for every open interval I on the circle. (Do you have a guess for $H^1(S^1, \underline{\mathbb{Z}}_X)$?)

Exercise 4 (Gluing sheaves; 12 points).
Let X be a topological space and $(U_i)_{i\in I}$ an open covering of X. Suppose $\mathscr{F}_i$ is a sheaf on U_i, $i \in I$,

$$\varphi_{ij}\colon \mathscr{F}_{i|U_i\cap U_j} \longrightarrow \mathscr{F}_{j|U_i\cap U_j}$$

is an isomorphism, $i, j \in I$, and the following conditions are satisfied:

i) $\forall i \in I : \varphi_i = \mathrm{id}_{\mathscr{F}_i}$,

ii) $\forall i, j, k \in I : \varphi_{ik|U_i\cap U_j\cap U_k} = \varphi_{jk|U_i\cap U_j\cap U_k} \circ \varphi_{ij|U_i\cap U_j\cap U_k}$.

Prove that there are a sheaf $\mathscr{F}$ on X and isomorphisms $\psi_i\colon \mathscr{F}_{|U_i} \longrightarrow \mathscr{F}_i$, $i \in I$, such that

$$\forall i, j \in I : \quad \psi_{j|U_i\cap U_j} = \varphi_{ij} \circ \psi_{i|U_i\cap U_j}.$$

Show that $\mathscr{F}$ is unique up to canonical isomorphism, i.e., if $\mathscr{F}'$ and ψ'_i, $i \in I$, is a different set of such data, then there is a unique isomorphism $\chi\colon \mathscr{F} \longrightarrow \mathscr{F}'$ with $\psi_i = \psi'_i \circ \chi_{|U_i}$, $i \in I$.