Problems on Algebraic Geometry

Summer 2021

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Problem Set 1

Due: Monday, April 26, 2021, 4pm

Exercise 1 (The Yoneda lemma; 3+3+3+3+3 points).
a) Let $\mathcal{C}$ be a category and $A \in \mathrm{Ob}(\mathcal{C})$. Show that

$$\begin{array}{rcl} h_A\colon \mathcal{C} & \longrightarrow & \underline{\mathrm{Sets}} \\ B & \longmapsto & \mathrm{Mor}_{\mathcal{C}}(A,B) \\ f\colon B \longrightarrow C & \longmapsto & (g\colon A \longrightarrow B) \longmapsto (f\circ g\colon A \longrightarrow C) \end{array}$$

is a covariant functor and that

$$\begin{array}{rcl} h^A\colon \mathcal{C} & \longrightarrow & \underline{\mathrm{Sets}} \\ B & \longmapsto & \mathrm{Mor}_{\mathcal{C}}(B,A) \\ f\colon B \longrightarrow C & \longmapsto & (g\colon C \longrightarrow A) \longmapsto (g\circ f\colon B \longrightarrow A) \end{array}$$

is a contravariant functor. These functors are called *representable functors*.
b) Let $F\colon \mathcal{C} \longrightarrow \underline{\mathrm{Sets}}$ be a (covariant) functor. Let A be an object of $\mathcal{C}$ and $a \in \mathcal{F}(A)$. Show that

$$\begin{array}{rcl} h_a\colon h_A & \longrightarrow & F \\ f\colon A \longrightarrow B & \longmapsto & F(f)(a) \end{array}$$

is a natural transformation of functors.
c) Let $\Phi\colon h_A \longrightarrow F$ be a natural transformation of functors. Define

$$a_\Phi := \Phi(A)(\mathrm{id}_A) \in F(A).$$

Prove that the assignments $\Phi \longmapsto a_\Phi$ and $a \longmapsto h_a$ are inverse to each other and, therefore, identify the natural transformations between h_A and F with the elements of $F(A)$.
d) Rewrite this result for contravariant functors.
e) Conclude that two objects A and B define isomorphic functors h_A and h_B (h^A and h^B) if and only if they are isomorphic.

Exercise 2 (Groups and their centers; 8 points).
Let $\underline{\mathrm{Grps}}$ be the category of groups and $\underline{\mathrm{Ab}}$ the category of abelian groups. For a group G, denote by

$$Z(G) := \left\{\, g \in G \,\middle|\, \forall h \in G : g\cdot h = h \cdot g \,\right\}$$

its center. Is it possible to extend the map

$$\begin{aligned}\mathrm{Ob}(\underline{\mathrm{Grps}}) &\longrightarrow \mathrm{Ob}(\underline{\mathrm{Ab}}) \\ G &\longmapsto Z(G)\end{aligned}$$

to a functor $F\colon \underline{\mathrm{Grps}} \longrightarrow \underline{\mathrm{Ab}}$?

Exercise 3 (Adjoint functors; 5+5 points).
Let $\underline{\mathrm{Top}}$ be the category of topological spaces, $\underline{\mathrm{Sets}}$ the category of sets, and $F\colon \underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Sets}}$ the forgetful functor.
a) Construct a functor $G\colon \underline{\mathrm{Sets}} \longrightarrow \underline{\mathrm{Top}}$, such that, for all $A \in \mathrm{Ob}(\underline{\mathrm{Top}})$ and all $B \in \mathrm{Ob}(\underline{\mathrm{Sets}})$,

$$\mathrm{Mor}_{\underline{\mathrm{Top}}}\big(A, G(B)\big) = \mathrm{Mor}_{\underline{\mathrm{Sets}}}(F(A), B).$$

The functor G is said to be *right adjoint* to F (and F *left adjoint* to G).
b) Construct a functor $H\colon \underline{\mathrm{Sets}} \longrightarrow \underline{\mathrm{Top}}$, such that, for all $A \in \mathrm{Ob}(\underline{\mathrm{Top}})$ and all $B \in \mathrm{Ob}(\underline{\mathrm{Sets}})$,

$$\mathrm{Mor}_{\underline{\mathrm{Top}}}(H(A), B) = \mathrm{Mor}_{\underline{\mathrm{Sets}}}\big(A, F(B)\big).$$

Here, G is left adjoint to F and F right adjoint to G.

Exercise 4 (The category of finite sets; 7 points).
Describe the skeleton of the category of finite sets.