Problems on Algebra I – Series 9

WS 2020/2021

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Due: Monday, January 25, 2021, 12pm

Exercise 1 (Modules over principal ideal domains; 4+6 points).
a) Let R be a principal ideal domain and M a finitely generated R-module without torsion. Show that any submodule N of M is free of finite rank.
b) Let R be a ring, such that, for every $n \in \mathbb{N}$, every submodule N of $R^{\oplus n}$ is free. Show that R is a principal ideal domain.
Hint. First show that R is an integral domain. Recall when ideals of R are free modules.

Exercise 2 (The Nakayama lemma I; 2+4+4 points).
Let $p \in \mathbb{N}$ be a prime number, $R := \mathbb{Z}_{\langle p \rangle}$ the localization of the ring of integers at the multiplicatively closed subset $S := \{ k \in \mathbb{Z} \mid p \nmid k \}$ and $\mathfrak{m} := p \cdot R$ the maximal ideal of R.
a) Show that $\mathbb{Q}$ is an R-module.
b) Prove that $\mathbb{Q} = \mathfrak{m} \cdot \mathbb{Q}$.
c) What does Part b) tell you about the Nakayama lemma?

Exercise 3 (The Nakayama lemma II; 8 points).
Let R be a local ring with maximal ideal $\mathfrak{m}$, M a finitely generated R-module, and $N \subset M$ a submodule with

$$M = \mathfrak{m} \cdot M + N.$$

Prove that $M = N$.

Exercise 4 (Krull's intersection theorem; 5+5+2 points).
Let T be the set of all pairs (I, f) where $I \subset \mathbb{R}$ is an open interval with $0 \in I$ and $f\colon I \longrightarrow \mathbb{R}$ is a C^∞-function. We call $(I, f), (J, g) \in T$ *equivalent*, if there exists an open interval $K \subset \mathbb{R}$ with $0 \in K \subset I \cap J$, such that

$$f_{|K} \equiv g_{|K}.$$

We let R be the set of equivalence classes.
a) Show that addition and multiplication of functions endows R with the structure of a commutative ring and that R is a local ring whose maximal ideal $\mathfrak{m}$ is the ideal generated by the equivalence class of $(\mathbb{R}, \mathrm{id}_{\mathbb{R}})$. (The ring R is called the *ring of germs of C^∞-functions at* 0.)
b) Investigate

$$\bigcap_{k \in \mathbb{N}} \mathfrak{m}^k.$$

Hint. Is there a C^∞-function f on a suitable open interval $0 \in I \subset \mathbb{R}$, such that f is not identically zero on any open interval around zero and $t \longmapsto f(t)/t^k$ is still a C^∞-function on I, for all $k \in \mathbb{N}$?
c) What does the example tell you about R and Krull's intersection theorem?