Problems on Algebra I – Series 8

WS 2020/2021

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Due: Monday, January 18, 2021, 12pm

Exercise 1 (The universal property of the direct sum; 5 points).
Let R be a ring, $(M_i)_{i\in I}$ a family of R-modules, and $\bigoplus_{i\in I} M_i$ its direct sum. Define, for $k \in I$,

$$\begin{aligned} j_k \colon M_k &\longrightarrow \bigoplus_{i\in I} M_i \\ m &\longmapsto (m_i)_{i\in I} \text{ with } m_i = \begin{cases} m, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}. \end{aligned}$$

Prove that $\bigoplus_{i\in I} M_i$ has the following universal property: Given an R-module N and a collection of homomorphisms $f_k \colon M_k \longrightarrow N$, $k \in I$, there is a unique homomorphism $f \colon \bigoplus_{i\in I} M_i \longrightarrow N$ with $f \circ j_k = f_k$, $k \in I$. In other words,

$$\mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i, N\Big) \cong \bigtimes_{i\in I} \mathrm{Hom}_R(M_i, N).$$

Exercise 2 (The universal property of the direct product; 5 points).
Let R be a ring, $(M_i)_{i\in I}$ a family of R-modules, and $\bigtimes_{i\in I} M_i$ its direct product. Define, for $k \in I$,

$$\begin{aligned} p_k \colon \bigtimes_{i\in I} M_i &\longrightarrow M_k \\ (m_i)_{i\in I} &\longmapsto m_k. \end{aligned}$$

Show that $\bigtimes_{i\in I} M_i$ has the following universal property: Given an R-module N and a collection of homomorphisms $f_k \colon N \longrightarrow M_k$, $k \in I$, there is a unique homomorphism $f \colon N \longrightarrow \bigtimes_{i\in I} M_i$ with $p_k \circ f = f_k$, $k \in I$. In other words,

$$\mathrm{Hom}_R\Big(N, \bigtimes_{i\in I} M_i\Big) \cong \bigtimes_{i\in I} \mathrm{Hom}_R(N, M_i).$$

Exercise 3 (The rank of a free module; 10 points).
Let R be a ring, $s, t \in \mathbb{N}$ natural numbers, and $\varphi \colon R^{\oplus s} \longrightarrow R^{\oplus t}$ an **isomorphism**. Prove that $s = t$.

Exercise 4 (Torsion modules; 10 points).
Let k be a field and $R := k[x]$. Let M be an R-module which is finite dimensional as k-vector space. Show that M is a torsion module.

Exercise 5 (Modules over polynomial rings; 6+4 points).

a) Let k be a field and $R := k[x]$. Show that the datum of an R-module M is the same as the datum of a k-vector space V endowed with a k-linear map $f: V \longrightarrow V$.

b) Let k be a field, $n \geq 2$, and $R := k[x_1, ..., x_n]$. How would you describe R-modules in terms of k-vector spaces and k-linear maps?