Problems on Algebra I – Series 7

WS 2020/2021

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Exercise 1 (A point of multiplicity 2; 8 points).
Let k be a field, and $a, b \in k$. We look at the map

$$\begin{array}{rcl} l\colon \mathbb{A}^1_k & \longrightarrow & \mathbb{A}^2_k \\ t & \longmapsto & (t, a \cdot t + b). \end{array}$$

It defines the surjective ring homomorphism

$$\begin{array}{rcl} l^{\#}\colon k[x,y] & \longrightarrow & k[t] \\ x & \longmapsto & t \\ y & \longmapsto & a \cdot t + b. \end{array}$$

Let $I \subset k[x,y]$ be an ideal. Its image $l^{\#}(I)$ is an ideal in $k[t]$ and therefore of the form $\langle f \rangle$, for some $f \in k[t]$.
Now, let

$$I := \langle\, x^2, x \cdot y \,\rangle$$

and verify that

$$l^{\#}(I) = \begin{cases} \langle t \rangle, & \text{if } b \neq 0 \\ \langle t^2 \rangle, & \text{if } b = 0 \end{cases}.$$

In contrast, if $J = \langle x \rangle$, then always $l^{\#}(J) = \langle t \rangle$. (Note that $V(I) = V(J)$.)
Remark. The set $V(f)$ describes the intersection of $l(\mathbb{A}^1_k)$ with $V(I)$. Computing with the ideals rather than with zero sets, we have "intersection multiplicities". The above example shows that $(0,0)$ occurs in the **scheme** defined by I with multiplicity two.

Exercise 2 (A primary decomposition; 8 points).
Let k be a field and $R := k[x,y,z]$. Set $\mathfrak{p}_1 := \langle x, y \rangle$, $\mathfrak{p}_2 := \langle x, z \rangle$, and $\mathfrak{m} := \langle x, y, z \rangle$.
a) Show that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals, while $\mathfrak{m}$ is maximal.
b) Let $I := \mathfrak{p}_1 \cdot \mathfrak{p}_2$. Show that

$$I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$$

and that this a minimal primary decomposition of I.
c) Which components are isolated and which are embedded?

Exercise 3 (A primary decomposition; 8 points).
We work in the ring $R = k[x_1, x_2, x_3, x_4]$, k a field. Show that

$$\begin{aligned} &\langle\, x_1x_2 - x_4, x_1x_3 - x_4, x_2x_3 - x_4 \,\rangle = \\ &= \langle\, x_1, x_2, x_4 \,\rangle \cap \langle\, x_1, x_3, x_4 \,\rangle \cap \langle\, x_2, x_3, x_4 \,\rangle \cap \langle\, x_1 - x_2, x_2 - x_3, x_1^2 - x_4 \,\rangle \end{aligned}$$

is a minimal primary decomposition.

Exercise 4 (An ideal without primary decomposition; 4+4+4+4 points).
In this exercise, we work in the ring $R := C^0([0,1])$ of continuous functions on the interval $[0,1] \subset \mathbb{R}$.
a) Show that, for a point $x \in [0,1]$,

$$\mathfrak{m}_x := \{ f \in R \mid f(x) = 0 \}$$

is a maximal ideal in R.
b) Let $\mathfrak{m} \subset R$ be a maximal ideal of R. Show that there exists a point $x \in [0,1]$ with $\mathfrak{m} = \mathfrak{m}_x$.
c) Let $\mathfrak{q} \subset R$ be a primary ideal. Show that there is a unique point $x \in [0,1]$ with $\mathfrak{q} \subset \mathfrak{m}_x$.
d) Conclude that the zero ideal $\langle 0 \rangle \subset R$ cannot be written as the intersection of finitely many primary ideals.