Problems on Algebra I – Series 6

WS 2020/2021

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Due: Monday, January 4, 2021, 12pm

Exercise 1 (The universal property of localization; 10 points).
Let R be a ring, $S \subset R$ a multiplicatively closed subset, and $\varphi\colon R \longrightarrow R_S$, $x \longmapsto x/1$, the canonical homomorphism to the localization. Show that the pair (R_S, φ) has the following universal property: For any ring T and any homomorphism $\psi\colon R \longrightarrow T$, such that $\psi(S) \subseteq T^\star$, there is a unique homomorphism $\psi_S\colon R_S \longrightarrow T$ with $\psi = \psi_S \circ \varphi$.

Exercise 2 (Irreducible sets; 9+6 points).
A topological space X is called *irreducible*, if it is non-empty, and, if X_1 and X_2 are closed subsets, such that $X = X_1 \cup X_2$, then $X_1 = X$ or $X_2 = X$. Let X be a topological space and Y a subset of X. Then, Y inherits a topology as follows: A subset $U \subset Y$ is said to be *open*, if there is an open subset $\widetilde{U} \subset X$ with $U = Y \cap \widetilde{U}$. We call a subset $Y \subset X$ *irreducible*, if it is irreducible with respect to the induced topology.
i) Let X be a Noetherian topological space and Z a **closed** subset. Show that there are irreducible closed subsets Z_1,...,Z_r, such that

- $Z = Z_1 \cup \cdots \cup Z_r$,
- $Z_i \not\subseteq Z_j$, for $i \neq j$.

Show also that these closed subsets are uniquely determined.
The sets Z_i, $i = 1, ..., r$, are called the *irreducible components* of Z.
ii) Let R be a Noetherian ring and $I \subset R$ an ideal. What is the relation between the primary decomposition of I and the above decomposition of the closed subset $V(I) \subset \operatorname{Spec}(R)$ into irreducible components?

Exercise 3 (A primary decomposition in $\mathbb{Z}[\sqrt{-5}]$; 3+3+4 points).
Define

$$\begin{aligned} \mathfrak{p}_1 &:= \langle 3, 2 + \sqrt{-5} \rangle \\ \mathfrak{p}_2 &:= \langle 3, 2 - \sqrt{-5} \rangle. \end{aligned}$$

i) Show that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals.
ii) Verify that the ideals $\mathfrak{p}_1^2$ and $\mathfrak{p}_2^2$ are coprime.
iii) Demonstrate that $\langle 9 \rangle = \mathfrak{p}_1^2 \cdot \mathfrak{p}_2^2 = \mathfrak{p}_1^2 \cap \mathfrak{p}_2^2$.

Exercise 4 (Primary ideals; 5 points).
Show the following: In the polynomial ring $\mathbb{Z}[t]$, a) the ideal $\mathfrak{m} = \langle 2, t \rangle$ is maximal and b) the ideal $\mathfrak{q} = \langle 4, t \rangle$ is $\mathfrak{m}$-primary, but c) $\mathfrak{q}$ is not a power of $\mathfrak{m}$.