Problems on Algebra I – Series 5

WS 2020/2021

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Exercise 1 (Maps between algebraic sets; 5+7 points).
Let k be a field and $Z \subset \mathbb{A}^n_k$ an algebraic set. Its *algebra of regular functions* is

$$k[Z] := k[x_1, ..., x_n]/I(Z).$$

Note that an element $f \in k[Z]$ defines indeed a function $f\colon Z \longrightarrow k$.
Let $W \subset \mathbb{A}^m_k$ and $Z \subset \mathbb{A}^n_k$ be algebraic sets and $F\colon W \longrightarrow Z$ a map. Write the induced map $F\colon W \longrightarrow Z \subset \mathbb{A}^n_k$ as $w \longmapsto (f_1(w), ..., f_n(w))$. We say that F is *regular*, if f_i is a regular function on W, $i = 1, ..., n$.
a) Let $F\colon W \longrightarrow Z$ be a regular map. Show that

$$\begin{aligned} F^\star\colon k[Z] &\longrightarrow k[W] \\ f &\longmapsto f \circ F \end{aligned}$$

is a homomorphism of k-algebras.
b) Suppose $\varphi\colon k[Z] \longrightarrow k[W]$ is a homomorphism of k-algebras. Show that there is a unique regular map $F\colon W \longrightarrow Z$ with $F^\star = \varphi$.

Exercise 2 (Closed points of the spectrum; 8 points).
Let R be a ring. A point $\mathfrak{p} \in \mathrm{Spec}(R)$ is *closed*, if $\{\mathfrak{p}\}$ is a Zariski closed subset of $\mathrm{Spec}(R)$.
Show that $\mathfrak{p} \in \mathrm{Spec}(R)$ is a closed point if and only if $\mathfrak{p}$ is a maximal ideal of R.

Exercise 3 (Descending chains of ideals in noetherian rings; 5 points).
Given an example of a noetherian ring R and a sequence $(I_k)_{k\in\mathbb{N}}$ of ideals in R, such that

$$\forall k \in \mathbb{N}: \quad I_k \supsetneq I_{k+1}.$$

Exercise 4 (Noetherian rings and spaces; 5+5+5 points).
a) Let k be a field. Then, one may define the polynomial ring $R := k[x_1, x_2, x_3, ...]$ in the infinitely many variables x_i, $i \geq 1$. Is R Noetherian?
b) A topological space X is called *Noetherian*, if it satisfies the descending chain condition for closed subsets, i.e., for any sequence

$$Z_1 \supset Z_2 \supset \cdots$$

of closed subsets of X, there is an index k_0, such that $Z_k = Z_{k_0}$, for every $k \geq k_0$. Let R be a Noetherian ring. Show that $\mathrm{Spec}(R)$ is a Noetherian topological space.
c) Give an example of a **non**-Noetherian ring R, such that $\mathrm{Spec}(R)$ consists of just one point (and is, therefore, Noetherian).