Problems on Algebra I – Series 4

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Due: Monday, December 7, 2020, 12pm

Exercise 1 (The spectrum of $\mathbb{Z}[x]$; 3+3+3+3+3 points).

The aim of this exercise is to determine all prime and maximal ideals of $\mathbb{Z}[x]$.
i) Show that a prime ideal $\mathfrak{p}$ which is **not** principal contains two **irreducible** polynomials f_1 and f_2 with $f_1 \nmid f_2$ and $f_2 \nmid f_1$.
ii) Explain why the greatest common divisor of f_1 and f_2 in $\mathbb{Q}[x]$ is 1, so that there are polynomials $g_1, g_2 \in \mathbb{Q}[x]$ with $f_1g_1 + f_2g_2 = 1$.
iii) Deduce from ii) that the intersection $\mathbb{Z} \cap \mathfrak{p}$ is non-zero and therefore of the form $\langle p \rangle$ for some prime number $p \in \mathbb{Z}$.
iv) Infer that a non-principal prime ideal $\mathfrak{p} \subset \mathbb{Z}$ is of the form $\langle p, f \rangle$ where $p \in \mathbb{Z}$ is a prime number and $f \in \mathbb{Z}[x]$ is a primitive polynomial of positive degree, such that its class $\overline{f} \in \mathbb{F}_p[x]$ is irreducible. Is such an ideal maximal?
v) Now, describe all prime and all maximal ideals of $\mathbb{Z}[x]$.
Remark. A picture of $\operatorname{Spec}(\mathbb{Z}[x])$ may be found in the books D. Mumford, *The red book of varieties and schemes*, and D. Eisenbud, J. Harris, *The geometry of schemes*.

Exercise 2 (The spectrum of a ring continued; 3+3+3 points).

Let R be a ring and $X := \operatorname{Spec}(R)$.
i) Show that, for an ideal $I \subset R$, one has $V(I) = V(\sqrt{I})$.
ii) For a subset $Z \subset X$, define the ideal[1]

$$I(Z) := \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}.$$

Show that $I(V(I)) = \sqrt{I}$ holds for every ideal $I \subset R$.
iii) Let $Z \subset X$ be a closed subset. Prove that $V(I(Z)) = Z$.

Exercise 3 (Algebraic subsets of the affine line; 6 points).

Let k be a field. Describe the Zariski open subsets of the affine line $\mathbb{A}^1_k$. (Recall that $k[x]$ is a principal ideal domain.)

Exercise 4 (The Zariski topology on the affine plane; 10 points).

Let k be an infinite field. As a set, we have $\mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k$. Check that the Zariski topology on $\mathbb{A}^2_k$ is not the product topology of the Zariski topologies on the factors $\mathbb{A}^1_k$. What happens for a finite field?
Remark. The relevant notions from topology may be found, e.g., in Section 2.1 of the book: G. Laures, M. Szymik, *Grundkurs Topologie*, Spektrum Akademischer Verlag GmbH, Heidelberg, 2009, x+242 pp.

[1] The intersection over an empty set of ideals is, by definition, the whole ring R.