Problems on Algebra I – Series 3

WS 2020/2021

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Due: Monday, November 30, 2020, 12pm

Exercise 1 (Basic open subsets; 5+2+3+5 points).
Let R be a ring and $X := \mathrm{Spec}(R)$. For $f \in R$, set $X_f := X \setminus V(\langle f \rangle)$.
i) Show that the X_f, $f \in R$, form a *basis* for the Zariski-topology, i.e., for every Zariski-open subset $U \subset X$, there is a subset $F \subset R$, such that

$$U = \bigcup_{f \in F} X_f.$$

Hint: For an ideal $\mathfrak{a} \subset R$, one has $\mathfrak{a} = \sum_{f \in \mathfrak{a}} \langle f \rangle$.
ii) Prove that $X_f \cap X_g = X_{f \cdot g}$, $f, g \in R$.
iii) Check that $X_f = X$ holds, if and only if f is a unit.
iv) Show that X is *quasi-compact*, i.e., every open covering of X possesses a **finite** subcovering.

Exercise 2 (Semilocal rings; 5 points).
Give an example of a semilocal ring with more than one maximal ideal.

Exercise 3 (Boolean rings; 3+4+3 points).
Let R be a Boolean ring. Set $X := \mathrm{Spec}(R)$.
i) Show that, for $f \in R$, the set X_f is both open and closed (in the Zariski topology).
ii) Let $f_1, ..., f_n \in R$ and

$$\mathfrak{a} := \langle f_1, ..., f_n \rangle := \langle f_1 \rangle + \cdots + \langle f_n \rangle.$$

Prove that $\mathfrak{a}$ is a principal ideal.
iii) Suppose $f_1, ..., f_n \in R$. Demonstrate that there is an element $f \in R$, such that

$$X_f = X_{f_1} \cup \cdots \cup X_{f_n}.$$

Exercise 4 (Chains of ideals in factorial rings; 10 points).
Let R be a **factorial** ring and

$$\langle r_1 \rangle \subset \langle r_2 \rangle \subset \cdots \subset \langle r_k \rangle \subset \langle r_{k+1} \rangle \subset \cdots$$

an ascending chain of principal ideals. Show that this sequence becomes *stationary*, i.e., there is an index $k_0 \in \mathbb{N}$, such that

$$\langle r_k \rangle = \langle r_{k_0} \rangle \quad \text{for all } k \geq k_0.$$