Problems on Algebra I – Series 2

WS 2020/2021

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Exercise 1 (Minimal prime ideals; 8 points).
Let $R \neq 0$ be a ring. Show that the set Σ of prime ideals of R has a minimal element with respect to inclusion.

Exercise 2 (Boolean rings; 4+3+3).
i) Let R be a ring such that every element $x \in R$ satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal $\mathfrak{p}$ of R is a maximal ideal.
ii) A ring R is called *Boolean*, if every element $x \in R$ verifies $x^2 = x$. Show that $2x = x + x = 0$ holds true for every element x in a Boolean ring R.
iii) Let $R \neq 0$ be a Boolean ring and $\mathfrak{p} \subset R$ a prime ideal. Show that $\mathfrak{p}$ is a maximal ideal and that $R/\mathfrak{p}$ is a field of two elements.

Exercise 3 (Prime ideals; 1+2+3+4 points).
Determine all prime and maximal ideals of the following rings: i) $\mathbb{R}$, ii) $\mathbb{Z}$, iii) $\mathbb{C}[x]$, and iv) $\mathbb{R}[x]$.

Exercise 4 (The spectrum of a ring; 2+3+3+4 points).
Let $R \neq 0$ be a ring. We define

$$\mathrm{Spec}(R) := \{\, \mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal} \,\}.$$

For an ideal $\mathfrak{a} \subset R$, we set

$$V(\mathfrak{a}) := \{\, \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subset \mathfrak{p} \,\}.$$

Establish the following properties:
i) $V(0) = \mathrm{Spec}(R)$, $V(R) = \varnothing$.
ii) Let $\mathfrak{a}_i$, $i \in I$, be a family of ideals in R. Their *sum* $\sum_{i\in I} \mathfrak{a}_i$ is the ideal of all linear combinations $\sum_{i\in I} a_i$ with $a_i \in \mathfrak{a}_i$, $i \in I$, almost all zero. Then,

$$V\Big(\sum_{i\in I} \mathfrak{a}_i\Big) = \bigcap_{i\in I} V(\mathfrak{a}_i).$$

iii) For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R,

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

Remark: Call a subset $Z \subset \mathrm{Spec}(R)$ *Zariski-closed*, if there is an ideal $\mathfrak{a} \subset R$ with $Z = V(\mathfrak{a})$ and a subset $U \subset \mathrm{Spec}(R)$ *Zariski-open*, if the complement $Z = \mathrm{Spec}(R) \setminus U$

is Zariski-closed. The above properties say:
i') The empty set and Spec(R) are Zariski-open.
ii') The union of an arbitrary family of Zariski-open subsets is Zariski-open.
iii') The intersection of two Zariski-open subsets is Zariski-open.
So,

$$\mathcal{T} := \{\, U \subset \mathrm{Spec}(R) \mid U \text{ is Zariski-open} \,\}$$

is a topology on Spec(R), the *Zariski-topology*.
iv) Let $f\colon R \longrightarrow S$ be a homomorphism of rings. Define

$$\begin{array}{rcl} f^{\#}\colon \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R) \\ \mathfrak{p} & \longmapsto & f^{-1}(\mathfrak{p}). \end{array}$$

Show that $f^{\#}$ is continuous in the Zariski-topology.