Problems on Algebra I – Series 1

WS 2020/2021

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Due: Monday, November 16, 2020, 12pm

Exercise 1 (The Weyl algebra; 2+3+7+3 points).
Let $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ be the set consisting of all $\mathbb{C}$-linear maps $\lambda\colon \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$.
i) Endow $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ with the structure of a ring.
ii) The elements

$$\begin{array}{rcl} \lambda_1\colon \mathbb{C}[x] & \longrightarrow & \mathbb{C}[x] \\ p & \longmapsto & \dfrac{\mathrm{d}}{\mathrm{d}x}(p) \end{array}$$

and

$$\begin{array}{rcl} \lambda_2\colon \mathbb{C}[x] & \longrightarrow & \mathbb{C}[x] \\ p & \longmapsto & x \cdot p \end{array}$$

belong to $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$.
Compute

$$\lambda_1 \cdot \lambda_2 - \lambda_2 \cdot \lambda_1.$$

iii) Show that there is a unique subring $R \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$, such that

- $\lambda_1, \lambda_2 \in R$,
- if $R' \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x])$ is a subring with $\lambda_1, \lambda_2 \in R'$, then $R \subset R'$.

This ring is called the *Weyl algebra*.
Show that any element of R may be written as a finite $\mathbb{C}$-linear combination of elements of the form

$$\lambda_1^{m_1} \cdot \lambda_2^{m_2}, \quad m_1, m_2 \in \mathbb{N}.$$

iv) Write the element

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_1 - 2 \cdot \lambda_2 \cdot \lambda_1 \cdot \lambda_2 + 3 \cdot \lambda_2^2 \cdot \lambda_1 - 5$$

as a finite $\mathbb{C}$-linear combination of elements of the form

$$\lambda_1^{m_1} \cdot \lambda_2^{m_2}, \quad m_1, m_2 \in \mathbb{N}.$$

Exercise 2 (Principal ideals; 5 points).
Give a concrete example of an ideal $I \subset \mathbb{Z}[x]$ which is not a principal ideal.
Of course, you have to justify that the ideal you found is not a principal ideal.

Exercise 3 (Units and nilpotent elements; 3+3+4 points).
i) Let R be a ring and $n \in R$ a nilpotent element. Show that $1 + n$ is a unit.
ii) Deduce that the sum $u + n$ of a unit $u \in R$ and a nilpotent element $n \in R$ is a unit.
iii) Describe the units of $\mathbb{Z}/\langle n \rangle$ for $n \geq 1$.

Exercise 4 (Units in polynomial rings; 10 points).
Let R be a ring and $f = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ a polynomial. Show that f is a unit if and only if a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent elements.
Instructions.

- For "$\Longleftarrow$", use the previous exercise.
- For "$\Longrightarrow$", let $g = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ be a polynomial with $fg = 1$. Prove by induction on r that

$$a_n^{r+1} b_{m-r} = 0, \quad r = 0, ..., m. \tag{1}$$

  To this end write $fg = c_0 + c_1 x + \cdots + c_{m+n} x^{m+n}$ and look at $a_n^{r+1} c_{m+n-r-1}$. Finally, deduce from (1) that a_n is nilpotent and conclude by induction on n.