Problems on Algebra I – Series 13 (last series)

WS 2020/2021

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Due: Monday, Febuary 22, 2021, 12pm

Exercise 1 (Krull's Höhensatz; 10 points).
Let R be a noetherian ring and $I \subsetneq R$ a proper ideal. We say that $I \subset \mathfrak{p}$ is a *minimal prime ideal containing* I, if the image $\overline{\mathfrak{p}}$ of $\mathfrak{p}$ in R/I is a minimal prime ideal of R/I. Assume that there are $s \in \mathbb{N}$ and $a_1, ..., a_s \in R$ with $\langle a_1, ..., a_s \rangle = I$. Prove that

$$\mathrm{ht}(\mathfrak{p}) \le s$$

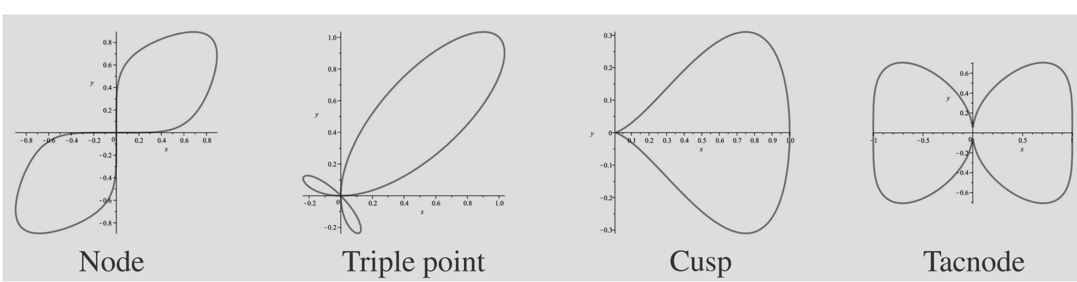
for every minimal prime ideal $\mathfrak{p} \subset R$ containing I.
What is the geometric interpretation of this statement?

Exercise 2 (Heights and dimension; 3+6+6 points).
Let k be a field, $A := k[w, x, y, z]/\langle w \cdot x - y \cdot z\rangle$, and $B := A/\langle x, y \rangle$. Set $\mathfrak{p} := \langle w, z \rangle \subset A$, and let $\mathfrak{q} \subset B$ be the image of $\mathfrak{p}$ under the natural homomorphism $A \longrightarrow B$.
a) Prove that $B \cong k[w, z]$ and that $\mathfrak{q}$ is a prime ideal of B.
b) Determine $\dim(A)$.
c) Prove that $\mathrm{ht}(\mathfrak{p}) = 1$ and $\mathrm{ht}(\mathfrak{q}) = 2$. What is the geometric interpretation of these results?

Exercise 3 (Singular and non-singular points; 6+9 points).
i) Compute the Jacobian for the following regular functions $f\colon \mathbb{C}^2 \longrightarrow \mathbb{C}$. Which points are non-singular by the "Jacobian criterion"? Find out which equation gives which curve in the following picture. Try to explain why the remaining points really are singular.

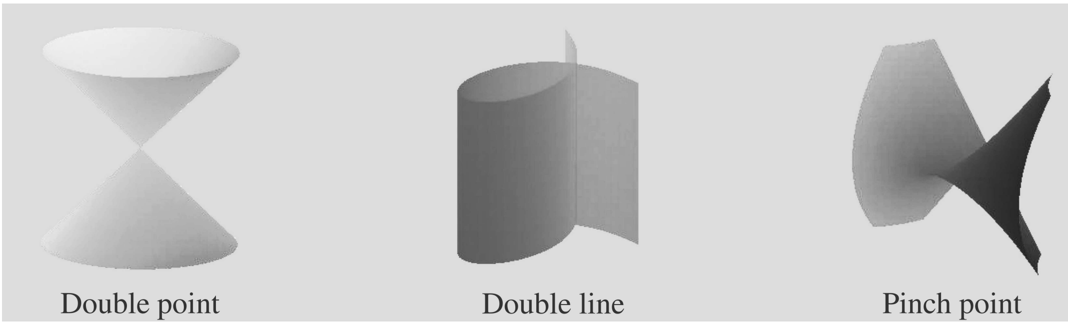
$$\text{a) } f(x,y) = x^2 - x^4 - y^4, \quad \text{b) } f(x,y) = xy - x^6 - y^6, \quad \text{c) } f(x,y) = x^3 - y^2 - x^4 - y^4,$$

$$\text{d) } f(x,y) = x^2y + xy^2 - x^4 - y^4.$$



ii) Do the same as in i) for the following regular functions $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$.

$$\text{a) } f(x,y,z) = xy^2 - z^2, \quad \text{b) } f(x,y,z) = x^2 + y^2 - z^2, \quad \text{c) } f(x,y,z) = xy + x^3 + y^3.$$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