Problems on Algebra I – Series 12

WS 2020/2021

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Exercise 1 (Dominant morphisms; 5+5 points).
Let k be an algebraically closed field. An *affine algebraic variety* is an **irreducible** algebraic set $X \subset \mathbb{A}_k^n$. Recall that an algebraic set $Z \subset \mathbb{A}_k^n$ is irreducible if and only if its coordinate algebra

$$k[Z] := k[x_1, ..., x_n]/I(Z)$$

is an integral domain.
A regular map $f: X \longrightarrow Y$ between algebraic varieties is *dominant*, if $f(X)$ is dense in Y.
a) Let $F: X \longrightarrow Y$ be a regular map between algebraic varieties and $F^\star: k[Y] \longrightarrow k[X]$ the corresponding homomorphism of algebras (Series 5, Exercise 1). Show that F is dominant if and only if $F^\star$ is injective.
b) Let X be an algebraic variety. The *function field* of X is the quotient field

$$k(X) := Q(k[X])$$

of the coordinate algebra $k[X]$ of X. Show that a dominant morphism $F: X \longrightarrow Y$ induces a field extension $F^\#: k(Y) \longrightarrow k(X)$, such that the diagram

$$\begin{array}{ccc} k[Y] & \xrightarrow{F^\star} & k[X] \\ \big\downarrow & & \big\downarrow \\ k(Y) & \xrightarrow{F^\#} & k(X) \end{array}$$

commutes.

Exercise 2 (Universal property of normalization; 10 points).
An affine algebraic variety X is said to be *normal*, if its coordinate algebra $k[X]$ is normal.
Let k be an algebraically closed field and X an affine algebraic variety over k. Show that there are a **normal** affine algebraic variety $\widetilde{X}$ and a **dominant** regular map

$$\nu: \widetilde{X} \longrightarrow X$$

which are **universal**, i.e., for every normal algebraic variety Z and every dominant regular map $\varphi: Z \longrightarrow X$, there is a unique regular map $\vartheta: Z \longrightarrow \widetilde{X}$ with $\varphi = \nu \circ \vartheta$:

$$\begin{array}{ccc} Z & \xrightarrow{\exists!\vartheta} & \widetilde{X} \\ & {\scriptstyle\varphi}\searrow & \big\downarrow{\scriptstyle\nu} \\ & & X. \end{array}$$

The pair $(\widetilde{X}, \nu)$ is the *normalization* of X.

Exercise 3 (Dimension I; 3+3+4 points).

Let k be a field, R a finitely generated k-algebra, and $\mathfrak{p}_1, ..., \mathfrak{p}_s$ the minimal prime ideals of R (see Theorem II.4.28, ii), in the lecture notes).

a) Assume that k is algebraically closed and that R is reduced. Write $R = k[x_1, ..., x_n]/I$ for a suitable natural number n and a suitable radical ideal $I \subset k[x_1, ..., x_n]$, and set $Z := V(I) \subset \mathbb{A}^n_k$. Define $\widetilde{\mathfrak{p}}_i \subset k[x_1, ..., x_n]$ as the preimage of $\mathfrak{p}_i$ under the projection $k[x_1, ..., x_n] \longrightarrow k[x_1, ..., x_n]/I$, $i = 1, ..., s$. What is the geometric significance of the varieties $V(\widetilde{\mathfrak{p}}_i) \subset \mathbb{A}^n_k$, $i = 1, ..., s$?

b) Set $R_i := R/\mathfrak{p}_i$, $i = 1, ..., s$. Prove that

$$\dim(R) = \max\Big\{ \mathrm{trdegk}(Q(R_i)) \,|\, i = 1, ..., s \Big\}.$$

c) Let $S \subset R$ be a subalgebra. Prove that $\dim(S) \leq \dim(R)$.

Hint. If $p_1, ..., p_s \in R$ are algebraically independent over k, then $k[p_1, ..., p_s] \subset R$ is an integral domain.

Exercise 4 (Dimension II; 3+5+2).

Set $R := \prod_{i=1}^{\infty} \mathbb{Z}/(2 \cdot \mathbb{Z})$.

a) Prove that R is not a noetherian ring.

b) Let $\mathfrak{p} \subset R$ be a prime ideal. Prove that the localization $R_\mathfrak{p}$ is a field.

Hint. Use the fact that $r^2 = r$ holds for every element $r \in R$.

c) Conclude that $\dim(R) = 0$.