Problems on Algebra I – Series 11

WS 2020/2021

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Due: Monday, Febuary 8, 2021, 12pm

Exercise 1 (Noether normalization; 10 points).
Let S be an integral domain and $R \subset S$ a subring, such that S is finitely generated as R-algebra. Prove that there are a non-zero element $f \in R \setminus \{0\}$, a natural number n, and elements $y_1, ..., y_n \in S$, such that

- ⋆ $y_1, ..., y_n$ are algebraically independent over R,[1]
- ⋆ the induced homomorphism $\varphi: R_f[y_1, ..., y_n] \longrightarrow S_f$ is an integral ring extension.

Exercise 2 (Normality of the ring of integers; 10 points).
Check that the ring of integers $\mathbb{Z}$ is normal.

Exercise 3 (Normal rings; 3+3+4 points).
a) Let R_1, R_2 be non-zero rings. Describe the spectrum of $R_1 \times R_2$ in terms of the spectra of R_1 and R_2. (Don't forget to think about the topology of the respective spaces.)
b) Let R_1 and R_2 be integral domains. Describe the total ring of fractions $Q(R_1 \times R_2)$ in terms of the quotient fields $Q(R_1)$ and $Q(R_2)$.
c) Use Part b) to construct a normal ring which is not an integral domain.

Exercise 4 (Separable polynomials; 4+6 points).
Let K be a field and $f \in K[x]$ an irreducible monic polynomial. Define the derivative f' of f by the usual rules of analysis.
a) Let $K \subseteq L$ be a field extension, such that f and f' split over L. Show that f is separable if and only if f and f' do not have a common zero in L.
b) Prove that f is inseparable, i.e., not separable, if and only if $f' = 0$.

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[1] This means that the homomorphism

$$\begin{array}{rcl} R[x_1, ..., x_n] & \longrightarrow & S \\ x_i & \longmapsto & y_i, \quad i = 1, ..., n, \end{array}$$

is injective.