Problems on Algebra I – Series 10

WS 2020/2021

M. Benyoussef, H. Lin, A. Schmitt

Due: Monday, Febuary 1, 2021, 12pm

Exercise 1 (Integral ring extensions I; 4+4+4 points).
a) Let $\varphi\colon R \longrightarrow S$ be an integral ring extension, $J \subset S$ an ideal, and $I := J^c$. Show that φ induces an integral ring extension $\overline{\varphi}\colon R/I \longrightarrow S/J$.
b) Let $\varphi\colon R \longrightarrow S$ be an integral ring extension and $T \subset R$ a multiplicatively closed subset. Verify that φ induces an integral ring extension $\varphi_T\colon R_T \longrightarrow S_T$.
c) Let R, S_1,...,S_n be rings and $f_i\colon R \longrightarrow S_i$, $i = 1, ..., n$, integral ring extensions. Show that

$$\begin{aligned} f\colon R &\longrightarrow \bigtimes_{i=1}^{n} S_i \\ x &\longmapsto (f_1(x), ..., f_n(x)) \end{aligned}$$

is also an integral ring extension.

Exercise 2 (Integral ring extensions II; 4+4 points).
Let R and S be integral domains and $\varphi\colon R \longrightarrow S$ an integral ring extension.
a) Suppose $f \in R$ is such that $\varphi(f)$ is a unit in S. Check that f is a unit in R.
b) Let $\mathfrak{n} \subset S$ be a maximal ideal and $\mathfrak{m} := \mathfrak{n} \cap R$. Show that $\mathfrak{m}$ is a maximal ideal in R.

Exercise 3 (Maximal ideals; 5 points).
Suppose that k is an algebraically closed field and that $\mathfrak{m} \subset k[x_1, ..., x_n]$ is a maximal ideal. Prove that there exists a point $(a_1, ..., a_n) \in k^n$ with

$$\mathfrak{m} = \langle\, x_1 - a_1, ..., x_n - a_n \,\rangle.$$

Exercise 4 (Study's lemma; 5 points).
Deduce the following result from the Nullstellensatz: Let k be an algebraically closed field and $f, g \in k[x_1, ..., x_n]$ polynomials. Assume that f is irreducible and $V(f) \subseteq V(g)$. Show that f divides g in $k[x_1, ..., x_n]$.

Exercise 5 (Algebraic sets over non-algebraically closed fields; 5+5 points).
Let k be a field which is not algebraically closed and $m, n \geq 1$ positive natural numbers.
a) Show that there is a polynomial $f \in k[x_1, ..., x_m]$, such that

$$V(f) = \{(0, ..., 0)\}.$$

b) Let $X \subset \mathbb{A}_k^n$ be an algebraic set. Prove that there is a polynomial $f \in k[x_1, ..., x_n]$, such that

$$V(f) = X.$$