A hierarchy of complex partial differential operators

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Abstract. The purpose of the present investigations is to built a homogeneous theory of complex partial differential equations of one complex variable of any order.

Potentials for the complex model partial differential operators $\partial_z^m \partial_{\overline{z}}^n, m, n \in \mathbb{N}_0$, serve to solve complex partial differential equations with these model operators as leading terms. Here $2\partial_{\overline{z}} = \partial_x + i\partial_y$ is the Cauchy-Riemann differential operator, $2\partial_z = \partial_x - i\partial_y$ its complex conjugate, $z = x + iy, \overline{z} = x - iy, x, y \in \mathbb{R}$. Particular model operators are the poly-analytic operator ∂_z^n, and the poly-harmonic operator $(\partial_z \partial_{\overline{z}})^n, n \in \mathbb{N}$, the latter being of importance for plane problems in mathematical physics.

Fundamental solutions for these model partial differential operators lead to potentials which serve to transform partial differential equations with these model operators as leading terms into singular integral equations to which in certain cases the Fredholm alternative do apply [1].

Within the last two decades the subject is in the process of developing [4, 2]. At presence mainly just model equations $\partial_z^m \partial_{\overline{z}}^n w = f$ are studied. Well-posed boundary value problems are determined and their solutions are given through proper integral representation formulas.

For the poly-analytic operator $\partial_{\overline{z}}^n, n \in \mathbb{N}$, the natural condition is the Schwarz boundary value problem

$$\mathrm{Re}\partial_{\overline{z}}^\mu w = \gamma_\mu (, \mathrm{Im}\partial_{\overline{z}}^\mu w(z_0) = c_\mu), 0 \le \mu \le n-1.$$

Dirichlet and Neumann problems and combinations of different boundary conditions are also available [2, 8].

For the poly-harmonic operator $(\partial_z \partial_{\overline{z}})^n, n \in \mathbb{N}$, there are many proper boundary value conditions. The higher the order the more boundary conditions are available. Basic are the Dirichlet

$$\partial_\nu^\mu w = \gamma_\mu, 0 \le \mu \le n-1,$$

the Neumann

$$\partial_\nu^\mu w = \gamma_\mu, 1 \le \mu \le n,$$

the Robin

$$(\alpha_\mu \partial_\nu^{\mu-1} + \beta_\mu \partial_\nu^\mu)w = \gamma_\mu, \alpha_\mu, \beta_\mu \in \mathbb{R}, \alpha_\mu^2 + \beta_\mu^2 \neq 0, 1 \leq \mu \leq n,$$

and the Riquier [9]

$$(\partial_z \partial_{\overline{z}})^\mu w = \gamma_\mu, 0 \leq \mu \leq n-1,$$

boundary condition. Here ∂_ν denotes the outward normal derivative on the boundary of the (regular) domain under consideration. But all kinds of combinations of these type of conditions are possible and this theory is not at all executed.

Having studied the poly-analytic and the poly-harmonic operators the general case $\partial_z^m \partial_{\overline{z}}^n, 0 \leq m \leq n$, can be factorized via $\partial_z^m \partial_{\overline{z}}^n = (\partial_z \partial_{\overline{z}})^m \partial_{\overline{z}}^{n-m}$.

Integral representation formulas for partial differential operators arise from applying the Gauss-Ostrogradski divergence theorem to a convolution of ∂w with some fundamental solution to ∂. With regard to boundary value problems fundamental solutions adjusted to the boundary conditions are important. Such fundamental solutions for the Cauchy-Riemann operator $\partial_{\overline{z}}$ are the Schwarz kernel and for the Laplacian $\partial_z \partial_{\overline{z}}$ the harmonic Green, Neumann, and Robin functions. An iteration of the Schwarz kernel provides a hierarchy of poly-analytic Schwarz kernels of any order [5] . Convolutions of Green, Neumann, and Robin functions lead to a variety of hybrid poly-harmonic Green functions [3, 6, 7].

For practical problems the knowledge of certain fundamental solutions in explicit form is important as in this case the integral representation formulas provide explicit solutions to the problem. For a large class of plane domains the parqueting-reflection principle [6] provides Schwarz, Green and Neumann kernel functions in explicit form. Their convolution produces higher order kernel functions.

The domains admissible to the parqueting-reflection principle are ones, the boundary of which consists of arcs from circles and straight lines such that the continued reflections of the domain at all these boundary parts provide a parqueting of the entire complex plane. Such domains are e.g. half planes, strips, plane sectors, circles, circle sectors, rectangles, certain triangles, hexagons, hyperbolic strips and half planes, lenses and moons, circular rings and ring sectors, etc. The parqueting-reflection principle does not always provide a harmonic Green function. E.g. ellipses can be reflected at their boundary resulting in a parqueting of the complex plane. However, no Green function is attained in this way. Instead the conformal invariance of the Green function has to be used. But in general the construction of conformal mappings onto the unit disc or the upper half plane is more involved than the application of the parqueting-reflection principle.

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