Advanced Statistical Physics
Problem Set 9

This is a Bonus problem set. Points obtained for these problems will contribute only to the numerator of your success ratio.

Suggested dates for two exams: Wednesday 16.02.2011 and Wednesday 23.02.2011, usual lecture time.

due: 05.01.2011
available at http://userpage.fu-berlin.de/psilvest/

Problem 9.1 Surface tension: Find the height of a liquid in a narrow vertical capillary of radius R. The density of the liquid is ρ and the surface tension $\mathcal{S}$. Consider both the case of perfect wetting and a finite wetting angle θ. **[2p]**

Problem 9.2 Thermodynamics: Use the method of determinants to show that

$$\kappa_T - \kappa_S = -TV \frac{\alpha_P^2}{C_P}, \tag{1}$$

where $\kappa_X = \frac{1}{V}\frac{\partial V}{\partial P}\Big|_X$, $\alpha_P = \frac{1}{V}\frac{\partial V}{\partial T}\Big|_P$ and C_P are the compressibility, the coefficient of thermal expansion and isobaric heat capacity. **[2p]**

Problem 9.3 Clapeyron-Clausius formula: The pressure of skates onto ice results in a decrease of the freezing point of water and in a melting of ice. Assume the mass of the skater to be 80 kg. The blade is touching the ice on a length of 20 cm and a width of 4 mm. The specific volumes of water and ice are $1.0 \cdot 10^{-3}\mathrm{m}^3/\mathrm{kg}$ and $1.1 \cdot 10^{-3}\mathrm{m}^3/\mathrm{kg}$ respectively. The latent heat of ice is $3.4 \cdot 10^{+5}\mathrm{J/kg}$. Calculate the decrease of the freezing point in the static case. Estimate whether this effect is sufficient to produce a film of water on which the skates can glide. **[2p]**

Problem 9.4 Microcanonical Ensemble: The Hamiltonian of $N \gg 1$ spins in a magnetic field B may be written as

$$H = BM = B \sum_{i=1}^{N} \mu s_i, \qquad (2)$$

where the projection of individual spins on the magnetic field axis may take two values $s_i = \pm 1/2$.

(a) Use the microcanonical ensemble to find the temperature T for a total energy of spins E and magnetic field B.

(b) Find the magnetic susceptibility

$$\chi = \left.\frac{\partial \langle M \rangle}{\partial B}\right|_T . \qquad (3)$$

[2p]

Problem 9.5 For those confused by problem 8.1: A dilute (Maxwell) gas is contained in a vessel at pressure P and temperature T. Calculate the rate at which the gas flows out of the vessel into a vacuum through a small hole of area A.

Find the distribution of the velocity component along the axis normal to the wall with the hole of the gas molecules leaving the vessel. **[2p]**

Problem 9.6 Thought experiment: A metallic cylinder is hanged on a throat. $N \gg 1$ electrons polarized along the z-axis enter the cylinder flying also along the z-axis. After the electrons stop in the cylinder, they would share a part of their momentum with the cylinder. However, a nonzero angular momentum of the cylinder $M_{cylinder}$ requires an energy of macroscopic rotation

$$E_{rotation} = \frac{M^2_{cylinder}}{2I}. \qquad (4)$$

Where would this energy come from? If the energy comes from $\cdots\cdots$ is not it in contradiction with the second law of Thermodynamics? **[2p]**

Problem 9.7 Find the adiabatic index γ ($PV^\gamma = const$ for adiabatic expansion of the gas) for a dilute vapor of C_2H_5OH at room temperature and at temperature of several thousand degrees Celsius. **[2p]**

Problem 9.8 Ortho- and Para-hydrogen: The two protons in the molecule H_2 have each spin 1/2. These spins, as you know from Quantum Mechanics, may be added into a total proton spin $j_p = s_{p1} + s_{p2} = 0$ or $j_p = s_{p1} + s_{p2} = 1$. The hydrogen molecule with $j_p = 0$ is called para-hydrogen. The molecule with $j_p = 1$ is called ortho-hydrogen and it may exist in three states with different projections $j_{pz} = +1, 0, -1$. The interaction of the proton spins with any magnetic field existing in the molecule is extremely weak and may be neglected. Thus at high temperatures the hydrogen gas is simply a mixture of 3/4 of ortho- and 1/4 of para-hydrogen.

Let us consider rotations of the hydrogen molecule. In both the ortho- and para- cases, the rotations are described by the same Hamiltonian

$$H = \frac{\hbar^2 l(l+1)}{2I}. \tag{5}$$

However, the symmetry of the molecule build from the two pairs of identical electrons and protons requires that the angular momentum l takes only even values $l_p = 0, 2, 4, ...$ for para-hydrogen and odd values $l_o = 1, 3, 5, ...$ for ortho-hydrogen.

Problem: A dilute hydrogen gas was cooled from room temperature to a very low temperature $T_0 \ll \hbar^2/I$. Flipping of the proton spins, which may lead to a transformation between ortho- and para-hydrogen molecules happens very rarely (takes hours). Thus after a fast cooling the gas is still a 3:1 mixture of ortho- and para- molecules. This mixture is kept in a constant volume and is thermally completely isolated for many hours, so that the numbers of ortho- and para- fractions have enough time to equilibrate. What will be the final temperature of the gas? **[2p]**

Problem 9.9 Consider a monoatomic gas at such large temperatures that ionization processes become important

$$A \rightarrow A^+ + e^-. \tag{6}$$

Let the fraction of ionized atoms be a known function of temperature, $\alpha(T)$ (it was calculated in the lecture, but you are not asked to use the explicit form of $\alpha(T)$). Find the heat capacity at a constant volume of the gas and plasma mixture $C_V(T)$. **[2p]**

Problem 9.10 Two chemistry students/alchemists were trying to produce ammonia (NH_3) by heating a mixture of Hydrogen and Nitrogen gases in a furnace. One student took 3 grams of H_2 and 14 grams of N_2 and after heating to the highest temperature and waiting till equilibrium was reached, he was able to produce 0.2 grams of ammonia. The second student took 14 grams of H_2 and 3 grams of N_2. How much ammonia would he produce if he uses the same vessel and the same furnace? **[2p]**

Problem 9.11* Find the spectrum of phonons in a chain of atoms with two different masses described by the Hamiltonian

$$H = \sum_{i=1}^{N} \left(\frac{p_{2i-1}^2}{2m} + \frac{p_{2i}^2}{2M} \right) + \frac{\mathcal{K}}{2} \sum_{j=1}^{2N} (q_{j+1} - q_j)^2. \tag{7}$$

Consider periodic boundary conditions (circle), $q_{i+2N} \equiv q_i$, $p_{i+2N} \equiv p_i$. **[2p]**

Can you guess, why the two branches of the spectrum are called "acoustic" and "optical" phonons?