Advanced Statistical Physics
Problem Set 6

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available at http://userpage.fu-berlin.de/psilvest/

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1. Extension of problem (3) from the previous set. Random matrices: As a model for energy levels of complex nuclei, Wigner considered $N \times N$ symmetric matrices whose elements are random. Let us assume that each element M_{ij} (for $i \geq i$) is an independent random variable taken from the PDF

$$p(M_{ij}) = \frac{1}{2a} \quad \text{for} \quad -a < M_{ij} < a, \quad \text{and} \quad p(M_{ij}) = 0 \quad \text{otherwise.} \tag{1}$$

(d) The eigenvalues of the random matrix M, λ_i ($i = 1, 2, \cdots, N$) also form a set of random variables with a certain PDF. The easy way to access the properties of this distribution is to consider the quantities $TrM^n = \sum \lambda_i^n$. Prove that λ_i are not independent random variables.

(One more hint, consider $\langle Tr(M^2) \rangle$ and $\langle (TrM)^2 \rangle$.) [2p]

2. Probabilities and Microcanonical ensemble: Consider random depositions of N 1-cent coins into M glasses. The Macrostate is specified by the numbers N and M. Individual microstates are described by the occupation numbers $(n_1, n_2, \cdots, n_M)$ of (indistinguishable) coins over (distinguishable) glasses. Find the number of microstates $\Omega(N, M)$ and the entropy $S(N, M)$.

(A standard approach to this problem is to consider the 1-dimensional chain of $N+M-1$ circles, N-white and $M-1$-black. Each subchain of n_i, $0 \leq n_i \leq N$, consecutive white circles models coins in one glass.) [2p]

Physical applications of this financial problem will be given in the next problem set.

3. Suppose now we have two "weakly interacting" subsystems: $N^{(1)}$ 1-cent coins and $N^{(2)}$ 2-cent coins, distributed over the same M glasses. Let any distribution $(n_1^{(1)}, n_2^{(1)}, \cdots, n_M^{(1)}, n_1^{(2)}, n_2^{(2)}, \cdots, n_M^{(2)})$ with $\sum_i (n_i^{(1)} + 2n_i^{(2)}) = N$ be a different microstate of the joint macrosystem (N, M).

(a) Find again the entropy $S(N, M)$.

(b) For $N \gg 1, M \gg 1$ find an equilibrium values of each sort of coins $N^{(1)*}$ and $N^{(2)*}$. By changing the parameters N and M, can we reach the situation in which in equilibrium:
1) Almost all the money will be kept in one sort of coins?
2) The same amounts of money will be stored in 1-cent and 2-cent coins?
3) There will be the same numbers of 1-cent and 2-cent coins? [2p]

4. Microcanonical Ensembe: Consider N classical 3-dimensional isotropic oscillators described by the Hamiltonian

$$\mathcal{H}(\{\vec{q}_i, \vec{p}_i\}) = \sum_{i=1}^{N} \left[\frac{\vec{p}_i^{\,2}}{2m} + \frac{m\omega^2 \vec{q}_i^{\,2}}{2} \right]. \tag{2}$$

(a) Calculate the entropy $S(E, N)$.
(b) Calculate the energy E and heat capacity C as functions of temperature T and N.
(c) Find the unconditional probability density $p(\vec{p}, \vec{q})$ for a single oscillator. [2p]

5. Answer the questions (b) and (c) of the previous problem for:
(a) N 2-dimensional oscillators described by the same Hamiltonian Eq. (2).
(b) N anisotropic oscillators

$$\mathcal{H}(\{\vec{q}_i, \vec{p}_i\}) = \sum_{i=1}^{N} \left[\frac{\vec{p}_i^{\,2}}{2m} + \frac{m\omega_x^2 x_i^2}{2} + \frac{m\omega_y^2 y_i^2}{2} + \frac{m\omega_z^2 x_i^2}{2} \right]. \tag{3}$$

In fact you do not need to redo any calculations for solving this problem. [2p]