Advanced Statistical Physics
Problem Set 5

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1. Rainbow singularity: Investigate the probability density function $p(\theta)$ for the rays of light reflected from the cloud of water droplets (rain). Here θ is the angle between the directions of incident and outgoing rays. The process of interest is when the ray of light is first refracted entering the surface of the raindrop, then reflected off the back of the drop, and again refracted as it leaves the drop. Refraction is described by the Snell's law, $n_1 \sin\phi_1 = n_2 \sin\phi_2$. Refractive index for water $n \approx 1.33$. Describe the singular behavior of $p(\theta)$. Explain, how the small frequency dependance of refractive index, $n = n(\omega)$, leads to appearance of the rainbow. [2p]

2. Cumulants: Use diagrammatic(graphical) method to calculate all (!) moments, $\langle x^m \rangle$ for the PDF(probability density function)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{x^2}{2\sigma^2}\right]. \tag{1}$$

Calculate also diagrammatically the moments of more general distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\lambda)^2}{2\sigma^2}\right]. \tag{2}$$

[2p]

3. Random matrices: As a model for energy levels of complex nuclei, Wigner considered $N \times N$ symmetric matrices whose elements are random. Let us assume that each element M_{ij} (for $i \geq j$) is an independent random variable taken from the PDF

$$p(M_{ij}) = \frac{1}{2a} \quad \text{for} \quad -a < M_{ij} < a, \quad \text{and} \quad p(M_{ij}) = 0 \quad \text{otherwise.} \tag{3}$$

(a) Calculate the characteristic (moment generating) function for each element M_{ij}.

(b) Calculate the characteristic function for the trace of the matrix, $T \equiv trM = \sum_i M_{ii}$.

(c) What does the central limit theorem imply about the PDF of the trace at large N? [2p]

4. Gold deposition: A mirror is plated by evaporating a gold electrode in vacuum by passing an electric current. The gold atoms fly off in all directions and a portion of them sticks to the glass (or to other gold atoms already on the glass plate). Assume that each column of deposited atoms is independent of neighboring columns, and that the average deposition rate is d layers per second.

(a) What is the probability of m atoms to be deposited at a site after a time t? What fraction of the glass is not covered by any gold atoms?

(b) What is the variance in the thickness? [2p]

5. Stirling's formula: Use the saddle point method to evaluate the value of $N!$ starting from the integral $\int x^N e^{-x} dx$. Calculate $\ln N!$ with the accuracy $\mathcal{O}(\frac{1}{N})$. Compare explicitly exact and approximate results for 10! and 20!. [2p]