Advanced Statistical Physics
Problem Set 3

due: 10.11.2010

Problem 3.1 Extension of problem 4. from the previous week.
Consider a capacitor (capacity $C = \varepsilon C_0$) filled with a dielectric whose dielectric constant depends on temperature, $\varepsilon = \varepsilon(T)$. The equation of state for the capacitor with a charge q is simply

$$V = \frac{q}{C(T)}. \quad (1)$$

The thermodynamic variables now are "force" $J = V$ and "displacement" $x = q$. In one experiment the capacitor is connected to a constant voltage source V. In second experiment the capacitor is first charged to the same voltage V, but than disconnected from the source. Find the difference of the heat capacity in these two cases, $c_q - c_V$. **[2p]**

Problem 3.2 Extension of problem 5. from the previous week.
A force J acting on an elastic filament is related to the displacement x via

$$J = ax - bT + cTx, \quad (2)$$

where a, b, c are constants. Furthermore the heat capacity at constant displacement is $C_x = A(x)T$.

(d) Calculate the heat capacity at constant tension, that is, $C_J = T\partial S/\partial T|_J$ as a function of T and J. **[2p]**

Problem 3.3 Jacobian transformations are a useful method of manipulation of thermodynamic derivatives. The Jacobian $\partial(u,v)/\partial(x,y)$ is defined as the determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}. \quad (3)$$

Prove the (mathematical) identities

(a)

$$\frac{\partial(u,v)}{\partial(x,y)} = -\frac{\partial(u,v)}{\partial(y,x)}, \quad (4)$$

(b)

$$\frac{\partial(u,y)}{\partial(x,y)} = \left.\frac{\partial u}{\partial x}\right|_y, \tag{5}$$

(c)

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(t,s)} \cdot \frac{\partial(t,s)}{\partial(x,y)}, \tag{6}$$

(d)

$$\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} = \frac{\partial(u,v)}{\partial(x,y)}. \tag{7}$$

You may find useful the algebraic identity, $\det A \cdot \det B = \det(AB)$. **[2p]**

$*$ Physical problems with Jacobians will follow in a week.

$*$ In mathematics we normally write a partial derivative as $\frac{\partial u}{\partial x}$ because it is always clear, what are the other variables in the function u. In thermodynamics, since one usually changes the variables many times back and forth, it is convenient to write $\left.\frac{\partial S}{\partial T}\right|_V$, or e.g. $\left(\frac{\partial S}{\partial T}\right)_{P,N}$. Be flexible and don't get confused.

Problem 3.4 Clapeyron Clausius formula: Find the change of the water boiling temperature from the ground floor to the top floor (621m) of the Burj Khalifa Tower, Dubai. The latent heat for boiling water is 2260kJ/kg (assume it to be constant). Density of the air $\approx$1.2kg/m^3. **[2p]**

Problem 3.5 Find analytically the temperature dependence of the pressure for the solid-vapor equilibrium. Approximate the vapor by an ideal gas and assume that the latent heat for sublimation is temperature independent. **[2p]**

Solution of the last problem from the previous week.

Problem 2.5 *A force J acting on an elastic filament is related to the displacement x via*

$$J = ax - bT + cTx, \tag{8}$$

where a, b, c are constants. Furthermore the heat capacity at constant displacement is $C_x = A(x)T$. Do at least one from the following:

(a) Calculate $\partial S/\partial x|_T$.

(b) Show that A should be independent of x, i.e. $dA/dx = 0$.

(c) Find $S(T, x) - S(0, 0)$.

(a) The Free energy

$$dF = -SdT + Jdx. \tag{9}$$

Thus

$$\left.\frac{\partial S}{\partial x}\right|_T = -\frac{\partial^2 F(T,x)}{\partial x \partial T} = -\left.\frac{\partial J}{\partial T}\right|_x = b - cx. \tag{10}$$

(b) We have $C_x = T\frac{\partial S}{\partial T} = A(x)T$, where $S = S(T, x)$. Thus

$$A = \left.\frac{\partial S}{\partial T}\right|_x, \tag{11}$$

and with the help of part (a) we find

$$\frac{dA}{dx} = \frac{\partial}{\partial x}\frac{\partial S}{\partial T} = \frac{\partial}{\partial T}\frac{\partial S}{\partial x} = \frac{\partial}{\partial T}(b - cx) = 0, \tag{12}$$

implying that A is independent of x.

(c) Since we already found the two derivatives of $S(T, x)$, we may write

$$S(T,x) = S(0,0) + \int_{T'=0}^{T'=T} \frac{\partial S(T,0)}{\partial T'} dT' + \int_{x'=0}^{x'=x} \frac{\partial S(T,x')}{\partial x'} dx' \tag{13}$$

$$= S(0,0) + \int_0^T A dT' + \int_0^x (b - cx')dx' \tag{14}$$

$$= S(0,0) + AT + bx - c\frac{x^2}{2}. \tag{15}$$