New infinite loop space operads

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This talk is a report on join work with:

Maria Basterra, Irena Bobkova, Kate Ponto, and Sarah Yeakel
Operads: the very basic idea

– codify operations

\[ \mu : A^n \rightarrow A \]

– study families of ’multiplications’

\[ \mu : \mathcal{V}(n) \times A^n \rightarrow A \]

– and how they fit together for different \( n \)

– codify notions ’up to homotopy’
The fundamental group $\pi_1(X, \ast)$

$\Omega X = \text{maps}([0, 1], \partial; X, \ast) = \text{loop space on } (X, \ast)$

multiplication defined by subdividing $[0, 1]$ into two pieces and running loops at a faster rate; associativity and inverses — up to homotopy!

From: William S. Massey
Higher homotopy groups $\pi_k(X,*)$

$\pi_k(X,*)$ is an abelian group for $k \geq 2$

$\Omega^k X = \text{maps}([0, 1]^k, \partial; X, *)$
$= \Omega(\Omega(\ldots \Omega(X, *) \ldots )) = k$-fold loop space on $(X, *)$

commutative – **up to homotopy**!

note: this is true already for $k = 2$
there is an obvious alternative homotopy;
are the homotopies homotopic? yes, for $\Omega^3(X)$;
are the homotopies of homotopies homotopic? yes, for $\Omega^4(X)$;

$\Omega^\infty$-spaces = homotopy everything spaces
Classically (1970s):

- to study operations up to homotopy
  (associative, commutative, ...)

- to give recognition principles for higher loop spaces,
  especially infinite loop spaces

Recall: \( \Omega^\infty \)-spaces = homotopy everything spaces

give rise to generalized (co)homology theories:
\( H_*(-; A), K^*(-), \pi^s_*(-), \Omega_*(-) \)

they have Mayer-Vietoris sequences; hence
they are computable by divide and conquer techniques!
Renaissance (1990s):

- to study new structures motivated by mathematical physics (TQFTs and CTFs, string theory, ...)
- moduli spaces and Segal’s category $\mathcal{M}$
- Ginzburg-Kapranov Koszul duality
- Deligne conjecture and Kontsevich formality
Fusion of classical and renaissance outlook:

Goal: prove that certain operads, such as studied in TQFT and CFT, are infinite loop space operads in the sense that they provide a completely different way of detecting infinite loop spaces and maps between them.

Main input: homology stability!

Homology stability has been a very active research area:
- mapping class groups and related groups
- topological moduli spaces of higher dim. manifolds
- moduli spaces of graphs
Content

Part I. Standard definitions and constructions
Part II. Operads with homology stability
Part III. Main theorem, examples and applications
Part IV. Proof
PART I
Operads and their algebras

A topological **operad** is a collection of spaces

\[ \mathcal{V} = \{\mathcal{V}(n)\}_{n \geq 0} \]

with \( \star \in \mathcal{V}(0) \), \( 1 \in \mathcal{V}(1) \), a right action of the symmetric group \( \Sigma_n \) on \( \mathcal{V}(n) \) and equivariant structure maps

\[ \gamma: \mathcal{V}(k) \times [\mathcal{V}(j_1) \times \ldots \times \mathcal{V}(j_k)] \to \mathcal{V}(j_1 + \ldots + j_k) \]

A **\( \mathcal{V} \)-algebra** is a based space \((X, \star)\) with equivariant structure maps

\[ \theta: \mathcal{V}(j) \times X^j \to X \]
If \((X, \ast)\) is a based space, the **free \(\mathcal{V}\)-space on** \(X\) is defined by

\[
\mathcal{V}(X) := \bigsqcup_{n \geq 0} \left( \mathcal{V}(n) \times \Sigma_n X^n \right) / \sim
\]

where \(\sim\) is a base point relation generated by

\[
(\sigma_i c; x_1, \ldots, x_{n-1}) \sim (c; s_i(x_1, \ldots, x_{n-1}))
\]

**Example:** note that in general \(\mathcal{V}(0) \neq \ast\)!

it defines a non-trivial example of a \(\mathcal{V}\)-algebra

**Morphisms** of operads and algebras - natural definition!
**Classical example:**

\[ C_n \text{ with } C_n(k) \subset \text{Emb}(\coprod_k D^n, D^n) \]

determined by midpoints and radii, and \( \simeq Conf_k(\mathbb{R}^n) \)

\[ \gamma : C_2(3) \times [C_2(2) \times C_2(3) \times C_2(4)] \rightarrow C_2(9) \]
Maps of operads:

\[ C_1 \to C_2 \to \cdots \to C_n \to \cdots \to C_{\infty} \]

**Example:** $\Omega^n(X)$ is a $C_n$-algebra

**Recognition Theorem:**
Connected (group-like) $C_n$-algebras are $\Omega^n$-spaces. More generally, the group completion of an $C_n$-algebra is an $\Omega^n$-space.

Stasheff, Broadman-Vogt, May, Barrett-Eccles, Milgram, ...
Rainer Vogt 1942-2015
CFT example:

\[ \mathcal{M} \text{ with } \mathcal{M}(n) = \coprod_{g \geq 0} \mathcal{M}_{g,n+1} \]

\[ \mathcal{M}_{g,n+1} = \text{moduli space of Riemann surfaces of genus } g \text{ with } n + 1 \text{ boundary components} \]

\[ \gamma : \mathcal{M}_{0,2+1} \times [\mathcal{M}_{0,2+1} \times \mathcal{M}_{0,0+1}] \to \mathcal{M}_{0,2+1} \]

Example: a CFT is a symmetric monoidal functor \( \mathcal{F} \) from Segal’s category of Riemann surfaces to another symmetric monoidal category \( \mathcal{L} \);
\( \mathcal{F}(S^1) \) is an \( \mathcal{M} \)-algebra when \( \mathcal{L} = \text{spaces} \)
Aside: Group completion

**Algebraic:** \( M \rightarrow \mathcal{G}(M) = \) Grothendieck group of \( M \)

Example: \( \mathbb{N} \rightarrow \mathcal{G}(\mathbb{N}) = \mathbb{Z} \)

**Homotopy theoretic:**
\( M \rightarrow \Omega BM = \) loop space of \( BM \)

- \( M = G \) a group \( \implies \Omega BG \simeq G \)
- \( M \) discrete \( \implies \Omega BM \simeq \mathcal{G}(M) \)

**Group Completion Theorem:**
Let \( M = \bigcup_{n \geq 0} M_n \) be a topological monoid such that the multiplication on \( H_*(M) \) is commutative. Then

\[
H_*(\Omega BM) = \mathbb{Z} \times \lim_{n \to \infty} H_*(M_n) = \mathbb{Z} \times H_*(M_\infty)
\]
PART II
Operads with homology stability OHS

Let $I$ be a discrete abelian monoid. An $I$-grading on an operad $\mathcal{V}$ is a decomposition

$$\mathcal{V}(n) = \bigoplus_{g \in I} \mathcal{V}_g(n)$$

for each $n$ so that:
- the basepoint $\ast$ lies in $\mathcal{V}_0(0)$,
- the $\Sigma_n$ action restricts to an action on each $\mathcal{V}_g(n)$
- the structure maps restrict to maps

$$\gamma : \mathcal{V}_g(k) \times [\mathcal{V}_{g_1}(j_1) \times \ldots \times \mathcal{V}_{g_k}(j_k)] \to \mathcal{V}_{g+g_1+\ldots+g_k}(\Sigma j_i).$$
Assume that for some $\mathcal{A}_\infty$-operad $\mathcal{A}$ there is a map of operads

$$\mu : \mathcal{A} \to \mathcal{V}$$

and the image $\mu \mathcal{A}(2) \subset \mathcal{V}_0(2)$ is path-connected.

**Definition OHS:**

$\mathcal{V}$ is said to be an operad with **homology stability** if the maps

$$D = \gamma(-; *, \ldots, *) : \mathcal{V}_g(n) \to \mathcal{V}_g(0)$$

are homology isomorphism in degrees $* < \phi(g)$ where $\phi$ goes to infinity as $g$ goes to infinity.
**Classical example:**

\[ \mu = incl : C_2 \to C_\infty \]
\[ C_\infty(n) \simeq E \Sigma_n \simeq * \text{ satisfies homology stability trivially} \]

**CFT example:**

\[ \mu = incl : C_2 \to \mathcal{M} \]
\[ \mathcal{M}_g(n) = \mathcal{M}_{g,n+1} \simeq B\text{Diff}(F_{g,n+1}; \partial) \simeq B\Gamma_{g,n+1} \]

[Harer]
\[ H_*(B\Gamma_{g,n+1}) \text{ is independent of } g \text{ and } n \text{ for } g \text{ large enough} \]
PART III

Main Theorem: [Basterra, Bobkova, Ponto, T., Yeakel]

If $\mathcal{V}$ is an OHS, group completion

$$\mathcal{G} : \mathcal{V} \text{- spaces} \longrightarrow \Omega^\infty \text{- spaces}$$

defines a functor from $\mathcal{V}$-spaces to infinite loop spaces

with a compatible $\Omega^\infty$-map

$$\mathcal{G}\mathcal{V}(\ast) \times \mathcal{G}X \rightarrow \mathcal{G}X.$$
Examples and applications

Non-orientable surfaces:

\[ \mathcal{K} \text{ with } \mathcal{K}(k) = \coprod_{g \geq 0} \mathcal{K}_{g,k+1} \]

\[ \mathcal{K}_g(k) = \mathcal{K}_{g,k+1} \simeq B\text{Diff}(N_{g,k+1}; \partial) \simeq BN_{g,k+1} \]

Homology stability: [Wahl]
\[ H_*(BN_{g,k+1}) \text{ is independent of } g, k \text{ for } g \text{ large enough} \]

Spin/Pin surfaces:

Homology stability: [Harer, Bauer, Randal-Williams]
Moduli space of graphs:

Graphs with $Graphs(k) = \coprod_{g \geq 0} Graphs_{g,k+1}$

$Graphs_{g,k+1} = \text{Culler-Vogtmann's Outer space of type } W_{g,k+1}
\simeq BHtEq(W_{g,k+1}; \partial)$

$W_{g,k+1} = \text{graph with } g \text{ circuits and } k+1 \text{ ends}$

Homology stability: [Hatcher-Vogtmann-Wahl]
$H_*(BHtEq(W_{g,k+1}; \partial))$ is independent of $g, k$ for $g$ large
Application: constructing $\Omega^\infty$-maps

the action on $H_1(N_{g,1}) = \mathbb{F}_2^g$ induces a representation

$$\rho : \mathcal{N}_{g,1} \longrightarrow \text{GL}_g \mathbb{F}_2$$

$\prod_{g \geq 0} B\mathcal{N}_{g,1} \simeq \mathcal{K}(\ast)$ is the free $\mathcal{K}$-algebra on $\ast$
$\prod_{g \geq 0} B\text{GL}_g \mathbb{F}_2$ is a $\mathcal{K}$-algebras via $\rho$

$\rho$ is a map of $\mathcal{K}$-algebras;
hence induces a map of $\Omega^\infty$-spaces:

$$\rho : \mathbb{Z} \times B\mathcal{N}_{\infty}^+ \simeq \Omega^\infty \text{MTO}(2) \longrightarrow \mathbb{Z} \times B\text{GL}(\mathbb{F}_2)^+ \simeq K(\mathbb{F}_2)$$
Application: detecting $\Omega^\infty$-spaces

$\tilde{N}_{g,1}$ is defined as an extention of $N_{g,1}$ by $H_1(N_{g,1}) = \mathbb{F}_2^g$

$X := \coprod_{g \geq 0} B\tilde{N}_{g,1}$ is an $K$-algebra

$\mathbb{Z} \times B\tilde{N}_\infty^+$ is an $\Omega^\infty$-space
PART IV

Sketch of Proof in 4 Steps

Step 1.: Replacement

Let $\mathcal{V}$ be an OHS. Then the product operad
\[ \tilde{\mathcal{V}} := \mathcal{V} \times C_\infty \]
is an OHS with compatible maps of operads
\[ \mathcal{V} \xleftarrow{\pi_1} \tilde{\mathcal{V}} \xrightarrow{\pi} C_\infty \]
So any $\mathcal{V}$-space is a $\tilde{\mathcal{V}}$-space.

W.l.o.g. assume there exists $\pi : \mathcal{V} \to C_\infty$
Step 2.: Group completion for free \( \mathcal{V} \)-algebras

For any based space \( X \),

\[
\tau \times \pi : \mathbb{V}(X) \longrightarrow \mathbb{V}(\ast) \times \mathbb{C}_\infty(X),
\]

induces a map of limits of filtration quotients

\[
F_n / F_{n-1} = \mathcal{V}_\infty(n) \times \Sigma_n X^\wedge n \longrightarrow \tilde{F}_n / \tilde{F}_{n-1} = \mathcal{V}_\infty \times \mathcal{C}_\infty(n) \times \Sigma_n X^\wedge n
\]

Because of the **homology stability assumption**, this is an \( H_* \)-isomorphism. Hence, by the Whitehead theorem,

\[
\mathcal{G}(\tau) \times \mathcal{G}(\pi): \mathcal{G}(\mathbb{V}(X)) \xrightarrow{\sim} \mathcal{G}(\mathbb{V}(\ast)) \times \mathcal{G}(\mathbb{C}_\infty(X))
\]

Recall: \( \mathcal{G}(\mathbb{C}_\infty(X)) \simeq \Omega^\infty \Sigma^\infty(X) \)
Step 3.: Functorial construction of $\Omega^\infty$-space

**Bar construction** for monad $\mathcal{V}$, $\mathcal{V}$-algebra $X$, and $\mathcal{V}$-functor $F$

$$B_\bullet(F, \mathcal{V}, X) := \{q \mapsto F(\mathcal{V}^q(X))\}$$

1. $|B_\bullet(GF, \mathcal{V}, X)| \simeq |GB_\bullet(F, \mathcal{V}, X)|$ for any functor $G$
2. $|B_\bullet(\mathcal{V}, \mathcal{V}, X)| \simeq X$
3. $|B(F, \mathcal{V}, \mathcal{V}(X))| \simeq F(X)$
4. If $\delta : \mathcal{V} \to \mathcal{W}$ is a natural transformation of monads, then $\mathcal{W}$ is a $\mathcal{V}$-functor and $B_\bullet(\mathcal{W}, \mathcal{V}, X)$ is a simplicial $\mathcal{W}$-algebra.

**Claim:** The assignment $X \mapsto |GB_\bullet(\mathcal{C}_\infty, \mathcal{V}, X)|$ defines a functor from $\mathcal{V}$-spaces to $\Omega^\infty$-spaces.
Proof:

\[
|G B \bullet (\mathbb{C}_\infty, \forall, X)| \quad \rightarrow \quad |G B \bullet (\mathbb{C}_\infty, \forall, Y)|
\]

\[
|G B \bullet (\Omega^\infty \Sigma^\infty, \forall, X)| \quad \rightarrow \quad |G B \bullet (\Omega^\infty \Sigma^\infty, \forall, Y)|
\]

\[
|G \Omega^\infty B \bullet (\Sigma^\infty, \forall, X)| \quad \rightarrow \quad |G \Omega^\infty B \bullet (\Sigma^\infty, \forall, Y)|
\]

\[
|\Omega^\infty B \bullet (\Sigma^\infty, \forall, X)| \quad \rightarrow \quad |B \bullet (\Omega^\infty \Sigma^\infty, \forall, Y)|
\]

\[
\Omega^\infty |B \bullet (\Sigma^\infty, \forall, X)| \quad \rightarrow \quad \Omega^\infty |B \bullet (\Sigma^\infty, \forall, Y)|
\]

We used the recognition principle for \(\mathbb{C}_\infty\)-algebras.
Step 4.: Restoring $\mathcal{G}(\mathcal{V}(\ast))$

**Claim:** For any $\mathcal{V}$-space $X$, there is a homotopy fibration sequence

$$
\mathcal{G}\mathcal{V}(\ast) \to |\mathcal{G}B\mathcal{V}(\mathcal{V}, \mathcal{V}, X)| \to |\mathcal{G}B\mathcal{V}(\mathcal{C}_\infty, \mathcal{V}, X)|.
$$

**Claim:** For any $\mathcal{V}$-space $X$, there are weak homotopy equivalences

$$
|\mathcal{G}B\mathcal{V}(\mathcal{V}, \mathcal{V}, X)| \xleftarrow{\sim} \mathcal{G}X \xrightarrow{\sim} |\mathcal{G}B\mathcal{V}(\mathcal{C}_\infty, \mathcal{V}, \mathcal{V}(\ast) \times X)|
$$

where $\mathcal{V}(\ast) \times X$ has the diagonal $\mathcal{V}$-space structure.