Towards an understanding of ramified extensions of structured ring spectra

Birgit Richter
Joint work with Bjørn Dundas, Ayelet Lindenstrauss

Women in Homotopy Theory and Algebraic Geometry
Structured ring spectra

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We are interested in commutative monoids (commutative ring spectra) and their algebraic properties.
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- Take your favorite commutative ring $R$ and consider singular cohomology with coefficients in $R$, $H^*(-; R)$. The representing spectrum is the Eilenberg-MacLane spectrum of $R$, $HR$. 
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- Topological complex K-theory, $KU_0(X)$, measures how many different complex vector bundles of finite rank live over your space $X$. You consider isomorphism classes of complex vector bundles of finite rank over $X$, $\text{Vect}_C(X)$. This is an abelian monoid wrt the Whitney sum of vector bundles. Then group completion gives $KU_0(X)$:
  
  $$KU_0(X) = \text{Gr}(\text{Vect}_C(X))$$
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- The homotopy groups of $KO$ are more complicated.

$$\pi_*(KO) = \mathbb{Z}[\eta, y, w^{\pm 1}]/2\eta, \eta^3, \eta y, y^2 - 4w, \quad |\eta| = 1, |w| = 8.$$
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The map that assigns to a real vector bundle its complexified vector bundle induces a ring map $c : KO \to KU$. Its effect on homotopy groups is $\eta \mapsto 0$, $y \mapsto 2u^2$, $w \mapsto u^4$. In particular, $\pi_*(-(KU))$ is a graded commutative $\pi_*(-(KO))$-algebra.
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Rognes '08: $KU$ is a $C_2$-Galois extension of $KO$. 

**Definition (Rognes '08) (up to cofibrancy issues...)**

A commutative $A$-algebra spectrum $B$ is a $G$-Galois extension, if $G$ acts on $B$ via maps of commutative $A$-algebras such that the maps $\alpha_i: A \to B$ and $\beta: B \wedge A \to \prod G B(\ast)$ are weak equivalences. This definition is a direct generalization of the definition of Galois extensions of commutative rings (due to Auslander-Goldman).
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Examples

As a sanity check we have:

\[ \text{Let } R \to T \text{ be a map of commutative rings and let } G \text{ act on } T \text{ via } R\text{-algebra maps. Then } R \to T \text{ is a } G \text{-Galois extension of commutative rings iff } HR \to HT \text{ is a } G \text{-Galois extension of commutative ring spectra.} \]

\[ \text{Let } Q \subset K \text{ be a finite } G \text{-Galois extension of fields and let } O_K \text{ denote the ring of integers in } K. \text{ Then } \mathbb{Z} \to O_K \text{ is never unramified, hence } H\mathbb{Z} \to H\mathbb{O}_K \text{ is never a } G \text{-Galois extension.} \]

\[ \mathbb{Z}[\sqrt{i}] \text{ is wildly ramified at } 2, \text{ hence } \mathbb{Z}[\sqrt{i}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{i}] \text{ is not isomorphic to } \mathbb{Z}[\sqrt{i}] \times \mathbb{Z}[\sqrt{i}]. \]

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Examples, continued

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Rognes ’08:

$$L_p \to KU_p$$

is a $C_{p-1}$-Galois extension. Here, the $C_{p-1}$-action is generated by an Adams operation.
Connective covers

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There are trace maps

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**BUT:** Trace methods work for **connective spectra**, these are spectra with trivial negative homotopy groups.
Connective spectra

For any commutative ring spectrum $R$, there is a commutative ring spectrum $r$ with a map $j: r \to R$ such that $\pi_*(j)$ is an isomorphism for all $* \geq 0$. BUT: A theorem of Akhil Mathew tells us, that if $A \to B$ is $G$-Galois for finite $G$ and $A$ and $B$ are connective, then $\pi_*(A)$ $\to$ $\pi_*(B)$ is étale. $\pi_*(ko) = \mathbb{Z}[\eta, y, w]/2\eta, \eta^3, \eta y, y^2 - 4w$ $\to$ $\pi_*(ku) = \mathbb{Z}[u]$ is certainly not étale. We have to live with ramification!
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Theorem (Dundas, Lindenstrauss, R)

$ko \to ku$ is wildly ramified.

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If we have a $G$-action on a commutative $A$-algebra $B$ and if $h: B \wedge_A B \to \prod_G B$ is a weak equivalence, then Rognes shows that the canonical map

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What is $THH^A(B)$? Topological Hochschild homology of $B$ as an $A$-algebra, i.e., $THH^A(B)$ is the geometric realization of the simplicial spectrum

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\cdots \xrightarrow{\wedge_A} B \wedge_A B \wedge_A B \xrightarrow{\wedge_A} B \wedge_A B$$
\[ THH^A(B) \] measure the ramification of \( A \to B \)!
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If $B$ is commutative, then we get maps

$$B \to \text{THH}^A(B) \to B$$

whose composite is the identity on $B$. 
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Thus $B$ splits off $THH^A(B)$. If $THH^A(B)$ is larger than $B$, then $A \to B$ is ramified.
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We abbreviate $\pi_*(THH^A(B))$ with $THH_*^A(B)$. 
The $ko \rightarrow ku$-case
The $ko \to ku$-case

**Theorem (DLR)**

- As a graded commutative augmented $\pi_\ast(ku)$-algebra

$$\pi_\ast(ku \wedge_{ko} ku) \cong \pi_\ast(ku)[\tilde{u}] / \tilde{u}^2 - u^2$$

with $|\tilde{u}| = 2$. 
The $ko \rightarrow ku$-case

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$$\pi_*(ku \wedge_{ko} ku) \cong \pi_*(ku)[\tilde{u}]/\tilde{u}^2 - u^2$$

with $|\tilde{u}| = 2$.

- The Tor spectral sequence

$$E^{2}_{*,*} = \text{Tor}_{*,*}^{\pi_*(ku \wedge_{ko} ku)}(\pi_*(ku), \pi_*(ku)) \Rightarrow \text{THH}^{ko}_{*}(ku)$$

collapses at the $E^2$-page.
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  $$\pi_\ast(ku \wedge_{ko} ku) \cong \pi_\ast(ku)[\tilde{u}]/\tilde{u}^2 - u^2$$

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- The Tor spectral sequence

  $$E^{2,2} = \text{Tor}_\ast,\ast^{\pi_\ast(ku \wedge_{ko} ku)}(\pi_\ast(ku), \pi_\ast(ku)) \Rightarrow \text{THH}^{ko}_\ast(ku)$$

  collapses at the $E^2$-page.

- $\text{THH}^{ko}_\ast(ku)$ is a square zero extension of $\pi_\ast(ku)$:

  $$\text{THH}^{ko}_\ast(ku) \cong \pi_\ast(ku) \rtimes \pi_\ast(ku)/2u\langle y_0, y_1, \ldots \rangle$$

  with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$. 
Comparison to $\mathbb{Z} \to \mathbb{Z}[i]$

The result is very similar to the calculation of

$HH_*(\mathbb{Z}[i]) = THH^{HZ}(HZ[i])$ (Larsen-Lindenstrauss):
Comparison to $\mathbb{Z} \rightarrow \mathbb{Z}[i]$

The result is very similar to the calculation of $HH_*(\mathbb{Z}[i]) = THH^H\mathbb{Z}(HZ[i])$ (Larsen-Lindenstrauss):

$$HH_*^\mathbb{Z}(\mathbb{Z}[i]) \cong THH_*^H\mathbb{Z}(HZ[i]) = \begin{cases} 
\mathbb{Z}[i], & \text{for } * = 0, \\
\mathbb{Z}[i]/2i, & \text{for odd } *, \\
0, & \text{otherwise.}
\end{cases}$$
Comparison to $\mathbb{Z} \to \mathbb{Z}[i]$

The result is very similar to the calculation of $HH_* (\mathbb{Z}[i]) = THH^{\mathbb{H}\mathbb{Z}} (H\mathbb{Z}[i])$ (Larsen-Lindenstrauss):

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Hence

$$HH_*^{\mathbb{Z}} (\mathbb{Z}[i]) \cong \mathbb{Z}[i] \times (\mathbb{Z}[i]/2i) \langle y_j, j \geq 0 \rangle$$

with $|y_j| = 2j + 1$. 
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Use an explicit resolution to get that the $E^2$-page is the homology of

$$\ldots \xrightarrow{0} \Sigma^4 \pi_\ast(ku) \xrightarrow{2u} \Sigma^2 \pi_\ast(ku) \xrightarrow{0} \pi_\ast(ku).$$

As $\pi_\ast(ku)$ splits off $THH(ko)$ the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

Use that the spectral sequence is one of $\pi_\ast(ku)$-modules to rule out additive extensions.

Since the generators over $\pi_\ast(ku)$ are all in odd degree, and their products cannot hit the direct summand $\pi_\ast(ku)$ in filtration degree zero, their products are all zero.
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Idea of proof for $ko \rightarrow ku$:
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Contrast to tame ramification

Consider an odd prime $p$ and

\[
\begin{array}{ccc}
\ell & \longrightarrow & ku(p) \\
\downarrow & & \downarrow \\
j & \downarrow & j \\
L & \longrightarrow & KU(p)
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\pi_\ast(\ell) = \mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u] = \pi_\ast(ku(p)), \quad v_1 \mapsto u^{p-1}
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- Sagave: The map $\ell \rightarrow k U_p$ is log-étale.
- Ausoni proved that the $p$-completed extension even satisfies Galois descent for $THH$ and algebraic K-theory:

$$
THH(k U_p)^{hC_{p-1}} \simeq THH(\ell_p), \quad K(k U_p)^{hC_{p-1}} \simeq K(\ell_p).
$$
Tame ramification is visible!

\[ \ell \to ku(p) \] behaves like a tamely ramified extension:
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**Theorem** (DLR)

\[ THH_\ell^{\ast}(ku(p)) \cong \pi_{\ast}(ku(p)) \ast \pi_{\ast}(ku(p))\langle y_0, y_1, \ldots \rangle / u^{p-2} \]

where the degree of \( y_i \) is \( 2pi + 3 \).
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\ell \to ku_{(p)} behaves like a tamely ramified extension:

**Theorem (DLR)**

\[ THH_{\ell}^*(ku_{(p)}) \cong \pi_{\ast}(ku_{(p)})_{\ast} \times \pi_{\ast}(ku_{(p)}) \langle y_0, y_1, \ldots \rangle / u^{p-2} \]

where the degree of \( y_i \) is \( 2pi + 3 \).

\( p - 1 \) is a \( p \)-local unit, hence no additive integral torsion appears in \( THH_{\ell}^*(ku_{(p)}) \).
Other important examples

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$\text{Tmf}_0(3) \to \text{Tmf}_1(3)$ is $C_2$-Galois (Mathew, Meier) and closely related to $E(2)^{hC_2} \to E(2)$.
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We can control certain quotient maps, e.g. $tmf_1(3)(2) \to ku(2)$. 

Open questions

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- How bad is $tmf_0(3) \to tmf_1(3)$?
- Can we understand the ramification for the extensions $BP\langle n\rangle^{hC_2} \to BP\langle n\rangle$ for higher $n$? Here, $\pi_\ast(BP\langle n\rangle) = \mathbb{Z}(p)[v_1, \ldots, v_n]$. 

$BP\langle 2 \rangle$ has commutative models at $p = 2, 3$ (Hill, Lawson, Naumann)

- Are $ku$, $ko$ and $\ell$ analogues of rings of integers in their periodic versions, i.e., $ku = \mathcal{O}_{KU}$, $ko = \mathcal{O}_{KO}$, $\ell = \mathcal{O}_L$?
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