On the K-theory of finitely generated projective modules over a spectral ring

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What is a Waldhausen category and its K-theory ?

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- any isomorphism is a w-cofibration, a morphism from $\mathbf{0}$ is a w-cofibration, and a pushout of a w-cofibration is w-cofibration
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A space $\Omega wS_{\bullet}C$ is called the Waldhausen K-theory of C, and denoted by $K(C)$.

Quillen K-theory versus Waldhausen K-theory

Example of Quillen$K$-theory (Quillen 1973)

A : commutative ring $\mathcal{E}$: the category of finitely generated projective over A Take monomorphisms as w-cofibrations and isomorphisms as w-equivalences. Then, $K(\mathcal{E})$ is defined.

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- $K(\mathcal{E})$ is related with other “important amounts” of algebraic variety
- Some properties of Quillen K-theory (cf. resolution theorem) fail in Waldhausen K-theory.
- Topological K-theory: for a space X, $K^{top}(X)$ is defined by groupcompletion of the category which consists of isomorphism classes of vector bundles on X.

Background

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An $\mathbb{E}_\infty$-ring is a spectrum equipped with commutative additive and multiplicative law up to coherent homotopy.

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- In 1970s, Adams and May defined this notion.
- In 2009, Lurie refine the notation of $\mathbb{E}_\infty$-ring by using ∞-operad.

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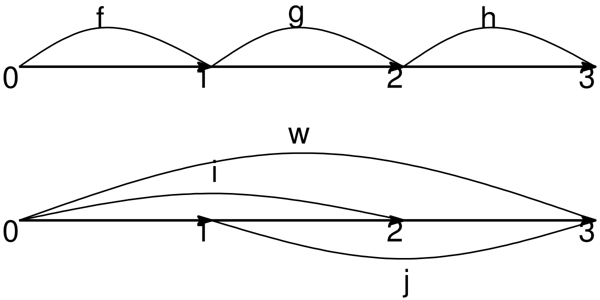

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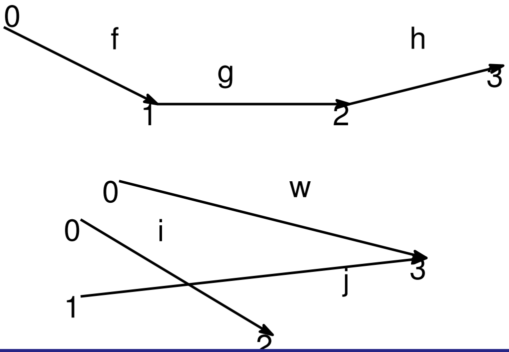

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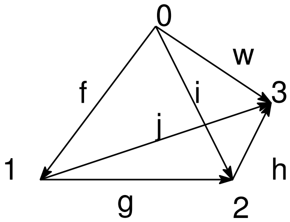

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The K-theory of a spectrum

(Elmendorf-Kriz-Mandell-May 1996) R : ring spectrum
fC_R : category of finite cell R-module spectra, then we define $K(R)$ by $K(fC_R)$.

- A w-cofibration is a Hurewicz cofibration (i.e. those morphisms which induce monomorphisms on the homotopy groups).
- A w-equivalence is a homotopy equivalence.

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Recent progress

- (EKMM 1996) R : ring spectrum fC_R : category of finite cell R-module spectra, then we define $K(R)$ by $K(fC_R)$. In this case, the functor $K(-)$ is represented by $\mathbb{Z} \times BGL^+$.
- (Morel-Voevodsky 1999) R : smooth scheme over a Noetherian scheme of finite Krull dimension Then, the mapping space of Motivic space $\pi_0 \mathrm{Map}(R,\ \Omega B(BGL)) \simeq K(R)$.

”Spectral ”algebraic geometry

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- (Lurie, 2012) By glueing $\mathbb{E}_\infty$-rings, the notion of “high schemes” and “structure sheaf” is defined.

Simplicial presheaves and simplicial sheaves are models of presheaves and hyper sheaves in the sense of derived algebraic geometry respectively.

Setting

R : connective $\mathbb{E}_\infty$-ring $\quad \mathbf{Mod}_R^{\infty proj}$: an ∞-category of projective R-modules of finite rank $\quad$ we define the algebraic K-theory of R by

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- (Barwick, Lurie) : They construct the algebraic K-theory of an ∞-category with the notion of "w-cofibrations". This time, we call the notion "w^∞-cofibrations".
- A w^∞-cofibration in $\mathbf{Mod}_R^{\infty proj}$ is a morphism whose cofiber lies in $\mathbf{Mod}_R^{\infty proj}$.

Problem

$\boldsymbol{R}$: connective $\mathbb{E}_\infty$-ring $\mathbf{Mod}_R^{\infty proj}$: an ∞-category of projective $\boldsymbol{R}$-modules of finite rank $\mathbf{CAlg}^{cn}$: the ∞-category of connective $\mathbb{E}_\infty$-rings $\mathbf{CAlg}^{\mathcal{G}}$: an ∞-category $\mathbf{CAlg}^{cn}$ equipped with either the Zariski topology or the Nisnevich topology $\mathcal{G}$ $\widehat{\mathcal{S}}$: the ∞-category of not-necessary small spaces $\mathcal{S}hv(\mathbf{CAlg}^{\mathcal{G}})$: the ∞-category of sheaves on $\mathbf{CAlg}^{\mathcal{G}}$

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Then, what is the explicit object which represents the sheaf on Zariski or Nisnevich ∞-topos $\mathcal{S}hv(\mathbf{CAlg}^{\mathcal{G}})$?

Main theorem

There is an equivalence of ∞-groupoids:

$$\mathbf{Map}_{\mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})}(\mathbf{Spec}^{\mathcal{G}} R,\ \Omega B^{\mathcal{G}}(B^{\mathcal{G}} GL)) \simeq \widetilde{K}^{\mathcal{G}}(\mathbf{Mod}_R^{\infty proj})$$

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- $\mathbf{Spec}^{\mathcal{G}}R$: an object in the essential image of Yoneda functor $\mathbf{CAlg}^{\mathcal{G}} \to \mathcal{S}hv(\mathbf{CAlg}^{\mathcal{G}})$.
- $\widetilde{(-)}^{\mathcal{G}}$: the sheafification from the ∞-category of functors on $(\mathbf{CAlg}^{\mathcal{G}})^{op}$ to $\mathcal{S}hv(\mathbf{CAlg}^{\mathcal{G}})$.
- $B^{\mathcal{G}}GL = \coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$: the coproduct of the classifying sheaf $B^{\mathcal{G}}GL_n$ of GL_n, where $B^{\mathcal{G}}$ is a functor given by taking classifying sheaf.
- $\Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)$: the group completion of $B^{\mathcal{G}}GL$.

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Zariski or Nisnevich sheafification of the algebraic$K$-theory is represented by an explicit object $\Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)$!!

The algebraic K-theory $K(\mathrm{Mod}_R^{proj})$

- $\mathbf{Mod}_R^{\infty proj}$ is not a stable ∞-category.
- Blumberg-Mandell and Barwick characterize the algebraic K-theory of the ∞-category of perfect R-modules and of truncated spectra respectively.

Outline of Proof

Definition of GL

- R : an $\mathbb{E}_\infty$-ring
- $\mathbf{CAlg}_R$: the ∞-category of R-algebras
- $\mathbf{Mod}_R$: an ∞-category of R-modules
- $\mathbf{Sym}^*_R$: Left adjoint of forgetful functor $\mathbf{CAlg}_R \to \mathbf{Mod}_R$

Set $M_{n,R} = \mathbf{Spec}\,\mathbf{Sym}_R \mathbf{End}_R(R^{\oplus n})$. This is a spectral monoid sheme which represents $R \mapsto End_R(R^{\oplus n})$.
Let $GL_{n,R}$ be a spectral group scheme obtained by inverting the determinant element in $\pi_0 M_{n,R}$. We denote GL_n by $GL_{n,\mathbb{S}}$, where $\mathbb{S}$ is the sphere spectrum.

Outline of Proof

R : connective $\mathbb{E}_\infty$-ring $\quad$ $\mathbf{Spec}^{\mathcal{G}}$ R : a spectral scheme
$B^{\mathcal{G}}GL$: a sheaf $\coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$.
For $I \subset J \subset \mathbb{N}$, we have the system
$\coprod_{i\in I} B^{\mathcal{G}}GL_i \to \coprod_{j\in J} B^{\mathcal{G}}GL_j$ given by inclusions. Then there is an equivalence of ∞-groupoids;

$$\mathbf{Map}_{\mathcal{S}hv(\mathbf{CAlg}^{\mathcal{G}})}(\mathbf{Spec}^{\mathcal{G}} R,\ B^{\mathcal{G}}GL) \simeq (\mathbf{Mod}_R^{\infty proj})^{\simeq},$$

where $(-)^{\simeq}$ denotes the maximal ∞-groupoid.

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- BGL_n is a hypercomplete flat sheaf.
- For finitely generated projective R-module M, we can choose $x_1, \cdots, x_m \in \pi_0 R$ such that each $M[x_i^{-1}]$ is a free $R[x_i^{-1}]$-module of finite rank n_i. Moreover, if M is projective of finite rank, we have $n_1 = \cdots = n_m$.

Outline of Proof

The adjoint functors, taking the classifying space functor B and taking the loop space functor Ω, induce a functor ΩB^{Π} satisfying the following commutative diagram

$$\begin{tikzcd}
Mon(\mathbf{Fun}((\mathbf{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \arrow[r, "\Omega B"] & Gp(\mathbf{Fun}((\mathbf{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \\
Mon(\mathbf{Fun}^{\Pi}((\mathbf{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \arrow[r, "\Omega B^{\Pi}"] \arrow[u] & Gp(\mathbf{Fun}^{\Pi}((\mathbf{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \arrow[u]
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We define the functor

$$\boldsymbol{\Omega B}^{\mathcal{G}} : Mon(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathbf{CAlg}^{\mathcal{G}})) \to Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathbf{CAlg}^{\mathcal{G}}))$$

by the composition of $\boldsymbol{\Omega B}^{\Pi}$ with
$i' : Mon(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathbf{CAlg}^{\mathcal{G}})) \to Mon(\mathbf{Fun}^{\Pi}((\mathbf{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}}))$.
This is the left adjoint of inclusion (i.e. groupcompletion).

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On the other hand, we have an equivalence induced by Yoneda embedding

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The sheafification of the objectwise group completion functor $R \mapsto \Omega B(B^{\mathcal{G}}GL(R))$ is equivalent to the group completion of the sheaf given by the assignment $R \mapsto (\Omega B^{\mathcal{G}}B^{\mathcal{G}}GL)(R)$. We have $(\Omega B^{\mathcal{G}}B^{\mathcal{G}}GL)(R) \simeq \Omega B(B^{\mathcal{G}}GL(R))$ after sheafification.

Thank you!