Weight filtration: some examples

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1. Smooth varieties

Let X be a smooth complex algebraic variety of dimension n. Let $j : \overline{X} \hookrightarrow X$ be a smooth compactification of X such that $D = \overline{X} - X$ is a normal crossings divisor. We may write $D = D_1 \cup \cdots D_N$ as the union of irreducible smooth divisors meeting transversally.
Let $D^{(0)} = X$ and for $0 < p \leq N$ let $D^{(p)}$ be the disjoint union of all p-fold intersections $D_{i_1} \cap \cdots \cap D_{i_p}$ with $\{i_1, \cdots, i_p\} \subset \{1, \cdots, N\}$. Since D is a normal crossings divisor, each $D^{(p)}$ is a smooth projective variety of dimension $n - p$.
The *weight spectral sequence* is given by (see [Del71]):

$$\boxed{E_1^{-p,q}(X) = H^{q-2p}(D^{(p)};\mathbb{Q}) \Longrightarrow H^{q-p}(X;\mathbb{Q}).}$$

The differential $d_1 : E_1^{-p,q}(X) \to E_1^{-p+1,q}(X)$ is defined by the sum of Gysin morphisms

$$i_*(j) : H^q(D_{i_1} \cap \cdots \cap D_{i_p}) \longrightarrow H^{q+2}(D_{i_1} \cap \cdots \cap \widehat{D}_{i_j} \cap \cdots \cap D_{i_p}).$$

This spectral sequence degenerates at the second stage, and it induces a filtration on the cohomology of X. The *weight filtration* W is defined by a shift of this filtration. We have $Gr_p^W H^{p+q}(X;\mathbb{Q}) \cong E_2^{p,q}(X)$. For all $n \geq 0$ the weight filtration satisfies

$$0 = W_{n-1}H^n(X;\mathbb{Q}) \subset W_n H^n(X;\mathbb{Q}) \subset \cdots \subset W_{2n}H^n(X;\mathbb{Q}) = H^n(X;\mathbb{Q}).$$

The picture looks like this:

$$E_1(X) = \left.\begin{array}{ccc}
\vdots & \vdots & \vdots \\
H^1(D^{(1)}) \xrightarrow{d_1} & H^3(D^{(0)}) \xrightarrow{d_1} & H^5(\overline{X}) \\
H^0(D^{(1)}) \xrightarrow{d_1} & H^2(D^{(0)}) \xrightarrow{d_1} & H^4(\overline{X}) \\
0 & H^1(D^{(0)}) \longrightarrow & H^3(\overline{X}) \\
0 & H^0(D^{(0)}) \longrightarrow & H^2(\overline{X}) \\
0 & 0 & H^1(\overline{X}) \\
0 & 0 & H^0(\overline{X}) \\
\hline
\end{array}\right| \Longrightarrow H(X) = \left.\begin{array}{ccc}
\vdots & \vdots & \\
Gr_5^W H^3 & Gr_5^W H^4 & Gr_5^W H^5 \\
Gr_4^W H^2 & Gr_4^W H^3 & Gr_4^W H^4 \\
Gr_3^W H^1 & Gr_3^W H^2 & Gr_3^W H^3 \\
0 & Gr_2^W H^1 & Gr_2^W H^2 \\
0 & 0 & Gr_1^W H^1 \\
0 & 0 & Gr_0^W H^0 \\
\hline
\end{array}\right|$$

Example 1.1. Let $X = \mathbb{C}^* \hookrightarrow \overline{X} = \mathbb{P}^1_{\mathbb{C}}$. Then $D = \{**\}$. We have:

$$E_1^{*,*}(X) = \begin{array}{c|c|}
2 & 1 \\ \hline
0 & 0 \\ \hline
0 & 1 \\ \hline
\end{array} \Longrightarrow H^*(X) = \begin{array}{c|c|}
1 & 0 \\ \hline
0 & 0 \\ \hline
0 & 1 \\ \hline
\end{array}$$

Since $E_2^{*,*}(X)$ is the cohomology of $X \simeq S^1$, the only non-trivial Gysin map of $E_1^{*,*}(X)$ must be onto. We find $Gr_2^W H^1(X;\mathbb{Q}) = \mathbb{Q}$.

Example 1.2 (Punctured Riemann surface)**.** Let $\overline{X} = R_g$ be a Riemann surface of genus g. Let $D \subset \overline{X}$ be a finite collection of p points. Consider the open variety $X = \overline{X} - D$. Then:

$$E_1^{*,*}(X) = \begin{array}{c|c|} p & 1 \\ \hline 0 & 2g \\ \hline 0 & 1 \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{c|c|} p-1 & 0 \\ \hline 0 & 2g \\ \hline 0 & 1 \\ \hline \end{array}$$

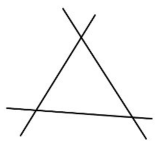
This gives: $Gr_1^W H^1(X;\mathbb{Q}) = \mathbb{Q}^{2g}$ and $Gr_2^W H^1(X) = \mathbb{Q}^{p-1}$.

Example 1.3. Let $X = \mathbb{C}^2 \setminus \{*\}$. The obvious way to compactify X is to embedd it in $\mathbb{P}^2_{\mathbb{C}}$. However, the complement $X - \mathbb{P}^2_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} \sqcup \{*\}$ is not a normal crossings divisor. Consider the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at the point $*$. This gives a smooth compactification $X \hookrightarrow \overline{X} = \widetilde{\mathbb{P}}^2_{\mathbb{C}}$ whose complement $D = \widetilde{\mathbb{P}}^2_{\mathbb{C}} - X = \mathbb{P}^1_{\mathbb{C}} \sqcup \mathbb{P}^1_{\mathbb{C}}$ is a normal crossings divisor. The non-trivial Betti numbers of $\widetilde{\mathbb{P}}^2_{\mathbb{C}}$ are $h^0 = 1$, $h^2 = 2$ and $h^4 = 1$. We have:

$$E_1^{*,*}(X) = \begin{array}{c|c|} 2 & 1 \\ \hline 0 & 0 \\ \hline 2 & 2 \\ \hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{c|c|} 1 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

Note that to compute the second page, we have two non-trivial Gysin morphisms. Since this spectral sequence computes the cohomology of $X \simeq S^3$, both morphisms must be onto. This gives $Gr_2^W H^3(X;\mathbb{Q}) = \mathbb{Q}$.

Example 1.4. Let D be a union of three complex projective lines in $\overline{X} = \mathbb{P}^2_{\mathbb{C}}$, with pairwise intersections.

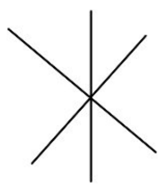


Let $X = \overline{X} - D$. Then $D^{(1)} = \mathbb{P}^1_{\mathbb{C}} \sqcup \mathbb{P}^1_{\mathbb{C}} \sqcup \mathbb{P}^1_{\mathbb{C}}$ and $D^{(2)} = \{* * *\}$. We have:

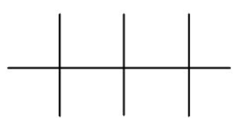
$$E_1^{*,*}(X) = \begin{array}{c|c|c|} 3 & 3 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 3 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{c|c|c|} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array}$$

This gives $Gr_2^W H^1(X;\mathbb{Q}) = \mathbb{Q}^2$ and $Gr_4^W H^2(X;\mathbb{Q}) = \mathbb{Q}$.

Example 1.5. Let X be the complement in $\mathbb{P}^2_{\mathbb{C}}$ of a union of three complex projective lines intersectiing at a point.



The complement $\mathbb{P}^2_{\mathbb{C}} - X$ is not a normal crossing divisor. We fix this by blowing-up $\mathbb{P}^2_{\mathbb{C}}$ at the intersection point. This gives a smooth compactification $X \hookrightarrow \overline{X} = \widetilde{\mathbb{P}}^2_{\mathbb{C}}$, where $D = \overline{X} - X$ is a union of four $\mathbb{P}^1_{\mathbb{C}}$'s intersecting in three distinct points.



Then: $D^{(1)} = \sqcup_4 \mathbb{P}^1_{\mathbb{C}}$ and $D^{(2)} = \{***\}$. We have:

$$E_1^{*,*}(X) = \begin{array}{|c|c|c|} 3 & 4 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 4 & 2 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{|c|c|c|} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array}$$

This gives $Gr_2^W H^1(X;\mathbb{Q}) = \mathbb{Q}^2$.

2. Singular projective varieties

Let X be a complex algebraic variety of dimension n. Let $X_\bullet \to X$ be a cubical hyperresolution of X, where X_α is smooth and of dimension $dim X_\alpha \leq n - |\alpha| + 1$ (see [GNAPP88]). This gives a weight spectral sequence

$$\boxed{E_1^{p,q}(X) = \bigoplus_{|\alpha|=p+1} H^q(X_\alpha;\mathbb{Q}) \Longrightarrow H^{p+q}(X;\mathbb{Q}).}$$

As before, the *weight filtration* is given by a shift of the filtration induced on $H^*(X;\mathbb{Q})$ by the weight spectral sequence. For all $n \geq 0$ it satisfies

$$0 = W_{-1}H^n(X;\mathbb{Q}) \subset W_0 H^n(X;\mathbb{Q}) \subset \cdots \subset W_n H^n(X;\mathbb{Q}) = H^n(X;\mathbb{Q}).$$

The picture looks like this:

$$E_1(X) = \begin{array}{|ccc} \vdots & \vdots & \vdots \\ H^4(X_{|1|}) & H^4(X_{|2|}) & H^4(X_{|3|}) \cdots \\ H^3(X_{|1|}) & H^3(X_{|2|}) & H^3(X_{|3|}) \cdots \\ H^2(X_{|1|}) & H^2(X_{|2|}) & H^2(X_{|3|}) \cdots \\ H^1(X_{|1|}) & H^1(X_{|2|}) & H^1(X_{|3|}) \cdots \\ H^0(X_{|1|}) & H^0(X_{|2|}) & H^0(X_{|3|}) \cdots \\ \hline \end{array} \Longrightarrow H(X) = \begin{array}{|ccc} \vdots & \vdots & \vdots \\ Gr_4^W H^4 & Gr_4^W H^5 & Gr_4^W H^6 \cdots \\ Gr_3^W H^3 & Gr_3^W H^4 & Gr_3^W H^5 \cdots \\ Gr_2^W H^2 & Gr_2^W H^3 & Gr_2^W H^4 \cdots \\ Gr_1^W H^1 & Gr_1^W H^2 & Gr_1^W H^3 \cdots \\ Gr_0^W H^0 & Gr_0^W H^1 & Gr_0^W H^2 \cdots \\ \hline \end{array}$$

For 2-cubical hyperresolutions the situation is much simpler: Let X be a complex algebraic variety and let $Y \subset X$ be the singular locus of X. By Hironaka's resolution of singularities exists a proper birrational morphism $f : \widetilde{X} \longrightarrow X$ where $\widetilde{X}$ is smooth and $\widetilde{Y} = f^{-1}(Y)$ is a normal crossings divisor in $\widetilde{X}$. We have a cartesian diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow f \\ Y & \xrightarrow[i]{} & X \end{array}$$

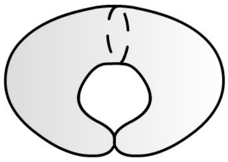
where f is an isomorphism outside Y and i is a closed immersion. This gives a Mayer-Vietoris exact sequence

$$\cdots \to H^k(X) \to H^k(\widetilde{X}) \oplus H^k(Y) \to H^k(\widetilde{Y}) \to H^{k+1}(X) \to \cdots.$$

If Y and $\widetilde{Y}$ are smooth, the above cartesian diagram is a cubical hyperresolution, and the weight spectral sequence is:

$$E_1^{*,*}(X) = \begin{array}{|l|l} H^2(\widetilde{X}) \oplus H^2(Y) & H^2(\widetilde{Y}) \\ \hline H^2(\widetilde{X}) \oplus H^2(Y) & H^2(\widetilde{Y}) \\ \hline H^2(\widetilde{X}) \oplus H^2(Y) & H^2(\widetilde{Y}) \\ \hline \end{array} \Longrightarrow H(X) = \begin{array}{|l|l} Gr_2^W H^2 & Gr_2^W H^3 \\ \hline Gr_1^W H^1 & Gr_1^W H^2 \\ \hline Gr_0^W H^0 & Gr_0^W H^1 \\ \hline \end{array}$$

Example 2.1 (Nodal curve)**.** Consider the complex curve $X = \{(x, y, z) \in \mathbb{P}^2; x^3 + y^3 = xyz\}$.



A resolution of X is given by the diagram

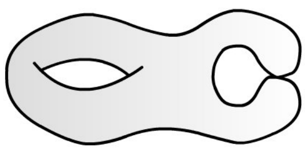
$$\begin{array}{ccc} \{**\} & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & X \end{array}$$

Then

$$E_1^{*,*}(X) = \begin{array}{|c|c} 1 & 0 \\ \hline 0 & 0 \\ \hline 2 & 2 \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{|c|c} 1 & 0 \\ \hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array}$$

This gives $Gr_0^W H^1(X; \mathbb{Q}) = \mathbb{Q}$ and $Gr_2^W H^2(X; \mathbb{Q}) = \mathbb{Q}$.

Example 2.2 (Riemann surface with nodes)**.** Let $\widetilde{X} = R_g$ be a Riemann surface of genus g. Let X be the same surface with p nodes as shown in the figure below.

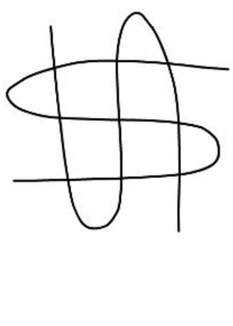


Then

$$E_1(X) = \begin{array}{|c|c} 1 & 0 \\ \hline 2g & 0 \\ \hline p+1 & 2p \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{|c|c} 1 & 0 \\ \hline 2g & 0 \\ \hline 1 & p \\ \hline \end{array}$$

This gives a non-trivial weight filtration on H^1: $Gr_0^W H^1(X; \mathbb{Q}) = \mathbb{Q}^p$ and $Gr_1^W H^1(X; \mathbb{Q}) = \mathbb{Q}^{2g}$.

Example 2.3 (Union of two Riemann surfaces)**.** Let X be the union of two generic cubic curves in $\mathbb{P}^2_{\mathbb{C}}$ (Riemann surfaces of genus 1). By Bézout, these two curves intersect at 9 distinct points.



A resolution for X is given by

$$\begin{array}{ccc} \{18 \text{ points}\} & \longrightarrow & \mathbb{T}^2 \sqcup \mathbb{T}^2 \\ \downarrow & & \downarrow \\ \{9 \text{ points}\} & \longrightarrow & X \end{array}$$

We have:

$$E_1(X) = \begin{array}{|c|c} 2 & 0 \\ \hline 4 & 0 \\ \hline 11 & 18 \\ \hline \end{array} \Longrightarrow H^*(X) = \begin{array}{|c|c} 2 & 0 \\ \hline 4 & 0 \\ \hline 1 & 8 \\ \hline \end{array}$$

Example 2.4 (Projective cone of a Riemann surface)**.** Let R_g be a Riemann surface of genus g. A resolution for its projective cone $X = P_cR$ is given by

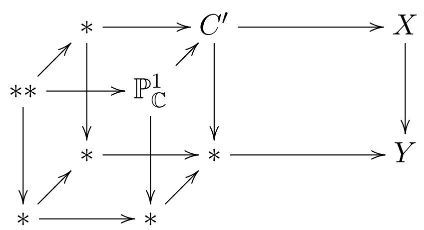
$$\begin{array}{ccc} R & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & P_cR \end{array}$$

Then

$$E_1(X) = \begin{array}{|c|c} 1 & 0 \\ \hline 2g & 0 \\ \hline 2 & 1 \\ \hline 2g & 2g \\ \hline 2 & 1 \\ \hline \end{array} \Longrightarrow H(X) = \begin{array}{|c|c} 1 & 0 \\ \hline 2g & 0 \\ \hline 1 & 0 \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$$

To compute the second page we used the fact that the Betti cohomology of P_cR is $h^1 = 0, h^2 = 1, h^3 = 2g, h^4 = 1$. The weight filtration is trivial: $H^k(P_cR)$ has pure weight k, for all $k \geq 0$.

Example 2.5 (Cusp singularity)**.** The following example is given by blowing down a nodal rational curve to a point. Let C be a nodal cubic curve in $\mathbb{P}^2_{\mathbb{C}}$. Choose $N \geq 12$ points in $\mathbb{P}^2_{\mathbb{C}}$ such that the proper transform C' of C has negative self-intersection. Let $X = Bl_N\mathbb{P}^2_{\mathbb{C}}$ and let Y be the blow-down of C' to a point inside X. Then Y is a projective surface with a normal isolated singularity (see Section 7 of [Tot14]). A cubical hyperresolution for Y is given by the diagram:



The weight spectral sequence for X is thus given by:

$$E_1^{r,*}(Y) = H^k(X) \oplus H^k(*) \oplus H^k(*) \longrightarrow H^k(\mathbb{P}^1_{\mathbb{C}}) \oplus H^k(*) \oplus H^k(*) \longrightarrow H^k(**)$$

$r = 0$ $\qquad r = 1$ $\qquad r = 2$

Therefore we have:

$$E_1(X) = \begin{array}{|c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline N+1 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline 3 & 3 & 2 \\ \hline \end{array} \Longrightarrow H(X) = \begin{array}{|c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline N & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \\ \hline\hline \end{array}$$

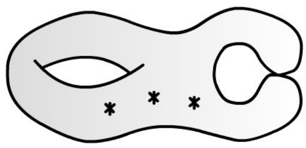
Hence $H^2(X)$ has a non-trivial weight filtration.

3. Singular non-compact varieties

This case is a mixture of the two previous cases. Let X be a complex variety. Let $X \hookrightarrow \overline{X}$ be a compactification of X. Then one may find a hyperresolution of the pair $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ in such a way that $D_\alpha = \overline{X}_\alpha - X_\alpha$ is a normal crossings divisor for each α. Then

$$\boxed{E_1^{-p,q}(X) = \bigoplus_\alpha E_1^{-p-|\alpha|+1,q}(X_\alpha) \Longrightarrow H^{q-p}(X;\mathbb{Q}).}$$

Example 3.1 (Singular punctured curve)**.** Let X be a Riemann surface of genus g with k punctures and one node. Let Z be the singular locus of X.



As in Example 2.2 we may resolve X by "sticking-off" the two points. Let $Y = R_g \setminus \{k \text{ points}\}$. Then We have:

$$E_1^{-p,q}(X) = E_1^{-p,q}(Y) \oplus E_1^{-p,q}(\{*\}) \oplus E_1^{-p-1,q}(\{**\}).$$

The term $E_1^{-p,q}(Y)$ is computed in example 1.2. We obtain:

$$E_1(X) = \begin{array}{c|c||c|} k & 1 & 0 \\ \hline 0 & 2g & 0 \\ \hline 0 & 2 & 2 \\ \hline\hline 0 & 0 & 0 \\ \hline \end{array} \Longrightarrow H(X) = \begin{array}{c|c||c|} k-1 & 0 & 0 \\ \hline 0 & 2g & 0 \\ \hline 0 & 1 & 1 \\ \hline\hline 0 & 0 & 0 \\ \hline \end{array}$$

Therefore the weight filtration on $H^1(X)$ has length 3:

$$Gr_0^W H^1(X) = \mathbb{Q},\ Gr_1^W H^1(X) = \mathbb{Q}^{2g},\ Gr_2^W H^1(X) = \mathbb{Q}^{k-1}.$$

References

[Del71] P. Deligne, *Théorie de Hodge. I*, Actes du Congrès International des Mathématiciens, Gauthier-Villars, Paris, 1971, pp. 425–430.

[GNAPP88] F. Guillén, V. Navarro-Aznar, P. Pascual, and F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, Lecture Notes in Mathematics, vol. 1335, Springer-Verlag, Berlin, 1988.

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