Spectral sequences and models

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1. Introduction

From Sullivan's theory, we know that the de Rham algebra of a manifold determines all its real homotopy invariants. In addition, the Formality Theorem of [1], exhibits the use of rational homotopy in the study of complex manifolds, in that it provides homotopical obstructions for the existence of Kähler metrics.

Bearing these results in mind, and with the objective to study complex homotopy invariants, Neisendorfer and Taylor [5] define the Dolbeault homotopy groups of a complex manifold by means of a bigraded model of its Dolbeault algebra of forms. Not only interesting in themselves, these new invariants prove to be useful in the computation of classical invariants such as the real homotopy or the cohomology of the manifold.

The Frölicher spectral sequence provides a connection between Dolbeault and de Rham models, and indicates an interplay between models and spectral sequences. In [3], Halperin and Tanré analyse this issue in the abstract setting, by constructing models of filtered cdga's and establishing a relationship with the bigraded minimal models of each stage of their associated spectral sequences. This allows the study of any spectral sequence coming from a filtration of geometric nature. The Dolbeault homotopy theory of Neisendorfer and Taylor fits naturally in this wider context.

As an application, Tanré studies in [6] the Borel spectral sequence associated with an holomorphic fibration, and constructs a Dolbeault model of the total space from those of the fiber and the base. Also, the filtered model of Halperin-Stasheff that controls the formality of a cdga fits into this context by means of the trivial filtration.

Our objective is to present the homotopy theory of filtered cdga's, focusing on its applications to the study of complex manifolds.

2. Homotopy theory of filtered cdga's

A *filtered cdga* (A, d, F) is a cdga (A, d) together with a decreasing filtration

$$0 \subseteq \cdots \subseteq F^{p+1}A \subseteq F^pA \subseteq \cdots \subseteq A,$$

such that the differential and the product are compatible with the filtration.

Any filtered cdga has an associated spectral sequence, each of whose stages is a bigraded differential algebra. Furthermore, every map of filtered cdga's compatible with filtrations induces a map between their respective spectral sequences. Such a map is an E_r*-quasi-isomorphism* if the induced map at the r-stage is a quasi-isomorphism of bigraded algebras. Every E_r-quasi-isomorphism is a quasi-isomorphism but the converse is not true in general.

In order to develop an homotopy theory for filtered cdga's, we generalize Sullivan's theory and introduce E_r-minimal models which we define step by step as follows. An E_r*-minimal extension* of degree n and weight p of a filtered cdga

(A,d,F) is a filtered cdga $A\otimes_d \Lambda(V)$, where V is a finite dimensional vector space of degree n and pure weight p, satisfying

$$dV \subset F^{p+r}(A^+\cdot A^+) + F^{p+r+1}A.$$

The filtration on $A\otimes\Lambda(V)$ is defined by multiplicative extension. All cdga's are augmented, and A^+ denotes the kernel of the augmentation. An *E_r-minimal cdga* is the colimit of a sequence of E_r-minimal extensions, starting from the base field. It follows that the differentials of the associated spectral sequence of an E_r-minimal cdga satisfy $d_0=\cdots=d_{r-1}=0$, and d_r is decomposable.

Theorem 1 (Halperin-Tanré)**.** *For every $r\geq 0$ and every filtered cdga (A,d,F) there exists an E_r-minimal model: that is an E_r-minimal cdga (M,D,F) together with an E_r-quasi-isomorphism $\psi:(M,D,F)\to(A,d,F)$. In particular, the induced map $E_r(\psi):(E_r(M),d_r)\to(E_r(A),d_r)$ is a bigraded model of $(E_r(A),d_r)$.*

Observe that for the trivial filtration, an E_0-minimal model is a Sullivan model, and so the above theorem can be viewed as a generalization of the classical theory.

The homotopical approach of [2] proves to be convenient in this situation. Define *r-homotopy equivalences* by means of a filtered path object $\Lambda(t,dt)$, with t of weight 0 and dt of weight r. Every E_r-minimal cdga M is *cofibrant*: any E_r-quasi-isomorphism $w:A\to B$ induces a bijection between classes of maps modulo r-homotopy equivalence, $w^*:[M,A]_r \xrightarrow{\sim} [M,B]_r$. In addition, E_r-minimal cdga's are *minimal*, in that every E_r-quasi-isomorphism between E_r-minimal cdga's is an isomorphism. The existence of E_r-minimal models endows the category of filtered cdga's with the structure of a *Sullivan category*. As a result, for all $r\geq 0$, we obtain an equivalence of categories

$$(E_r\text{-}\mathsf{Min}/\sim_r) \longrightarrow \mathrm{Ho}_r\,(\mathbf{F}\mathsf{CDGA}) = \mathbf{F}\mathsf{CDGA}[\mathcal{E}_r^{-1}],$$

between the quotient category of E_r-minimal cdga's modulo r-homotopy equivalence, and the localized category of filtered cdga's with respect to E_r-quasi-isomorphisms.

This provides a way to derive the functor of filtered indecomposables with respect to E_r-quasi-isomorfisms, obtaining a well defined notion of *E_r-homotopy groups*, as a new family of invariants for filtered cdga's.

Also, we have the following filtered version of formality. We say that a filtered cdga (A,d,F) is *E_r-formal* if there is a chain of E_r-quasi-isomorphisms

$$(A,d) \xleftarrow{\sim} \cdots \xrightarrow{\sim} (E_{r+1}(A),d_{r+1}).$$

3. Applications

We next present some applications of the homotopy theory of filtered cdga's to the study of complex manifolds.

Dolbeault homotopy. Let X be a complex manifold. Its complex de Rham algebra of $\mathcal{C}^\infty$ differential forms admits a bigrading by forms of type (p,q),

$$\mathcal{A}_{dR}(X) = \bigoplus \mathcal{A}^{p,q}(X).$$

The differential decomposes as $d = \partial + \overline{\partial}$. The *Frölicher spectral sequence* is the spectral sequence associated to $\mathcal{A}_{dR}(X)$, with the filtration defined by the first degree. Its 0-stage is the Dolbeault algebra $(E_0, d_0) = (\mathcal{A}^{*,*}(X), \overline{\partial})$, and its 1-stage is the Dolbeault cohomology $E_1 = H^{*,*}_{\overline{\partial}}(X)$. It converges to the de Rham cohomology $H^*_{dR}(X)$. The following result is a direct consequence of Theorem 1 applied to the Frölicher spectral sequence, and taking $r = 0$.

Theorem 2 (Neisendorfer-Taylor)**.** *There exists a de Rham model (M_X, D) of X together with a filtration such that $(E_0(M_X), d_0)$ is a Dolbeault model of X.*

In particular, given a Dolbeault model, one can build a de Rham model by defining a perturbation of its differential. If $E_1(\mathcal{A}_{dR}(X)) = E_\infty(\mathcal{A}_{dR}(X))$, then $\mathcal{A}_{dR}(X)$ is E_0-formal if and only if the manifold X is *strictly formal* in the sense of [5]. In particular, compact Kähler manifolds are E_0-formal.

Fibrations. Consider an holomorphic fibration $X_0 \to X \to Y$ of compact, connected, nilpotent complex manifolds. Assume as well that X_0 is Kähler, and that $\pi_1(Y)$ acts trivially on $H(X_0)$. In [4], Borel constructs a filtration of the Dolbeault algebra of X such that its associated spectral sequence converges to $H^{*,*}_{\overline{\partial}}(X)$, and $E_1 = (\mathcal{A}_{dR}(Y), \overline{\partial}) \otimes H_{\overline{\partial}}(X_0)$. The following result is a consequence of Theorem 1 applied to the Borel spectral sequence, with $r = 1$.

Theorem 3 (Tanré)**.** *With the previous assumptions, there exists a Dolbeault model M_X of X together with a filtration such that $(E_1(M_X), d_1) = (M_Y, \overline{\partial}) \otimes M_H$, where $(M_Y, \overline{\partial})$ is a Dolbeault model of Y and M_H is a Sullivan model of $H^*(X_0; \mathbb{C})$.*

Therefore a Dolbeault model for X can be built by taking the tensor product of a model of $H^*(X_0; \mathbb{C})$ by a Dolbeault model of Y, and defining a perturbation of the differential. An interesting application to the above result concerns compact connected Lie groups of even dimension.

Theorem 4 (Tanré)**.** *Let $T \to G \to G/T$, where G is a compact connected Lie group of even dimension, and T a maximal torus. Then G is Dolbeault formal if and only if the Frölicher spectral sequence satisfies $E_2(G) = E_\infty(G)$.*

This result facilitates finding examples of compact complex manifolds whose spectral sequence does not collapse at the second stage.

References

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