

# A SHORT COURSE ON THE INTERACTIONS OF RATIONAL HOMOTOPY THEORY AND COMPLEX (ALGEBRAIC) GEOMETRY

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ABSTRACT. These notes were written for the Summer School on “Rational Homotopy Theory and its Interactions” (July 2016, Rabat, Morocco). The objective of this course is to do several computations in rational homotopy theory for topological spaces that carry extra structure coming from complex geometry. We review results on the rational homotopy theory of complex manifolds, compact Kähler manifolds and (singular) complex projective varieties.

## 1. INTRODUCTION

A central construction in rational homotopy theory is Sullivan’s algebra of rational piece-wise linear forms. This is a commutative differential graded algebra (cdga for short)  $\mathcal{A}_{pl}^*(X)$  over  $\mathbb{Q}$ , defined for every topological space  $X$ , such that  $H^*(\mathcal{A}_{pl}^*(X)) \cong H^*(X; \mathbb{Q})$ . This algebra plays the role of the de Rham algebra of differential forms of a manifold and encodes more topological properties of the space than the cohomology ring. For instance, it contains Massey products. In general, this algebra is very large and difficult to deal with. The solution is to replace this algebra by a *minimal model*: a free cdga  $\Lambda(V)$  whose differential  $d$  is decomposable (a sum of products of positive elements), together with a quasi-isomorphism of cdga’s  $(\Lambda(V), d) \xrightarrow{\sim} \mathcal{A}_{pl}^*(X)$ . The minimal model can be understood as the smallest possible sub-cdga with the same cohomology and turns out to be a very powerful computational tool, as it contains all the rational homotopy information of the space.

A particularly useful situation to compute the minimal model is when the space  $X$  is *formal*: there is a string of quasi-isomorphisms from  $\mathcal{A}_{pl}^*(X)$  to its rational cohomology ring considered as a cdga with trivial differential. If  $X$  is formal then its rational homotopy type is completely determined by its cohomology ring, and higher order Massey products vanish. Examples of formal spaces include spheres and their products,  $H$ -spaces, symmetric spaces, and compact Kähler manifolds. In these notes we study properties of minimal models and aspects of formality for topological spaces carrying extra-structure of complex-geometric origin.

The complex de Rham algebra of forms of every complex manifold admits a decomposition into complex-valued forms of type  $(p, q)$  and its differential decomposes as  $d = \bar{\partial} + \partial$ . The interplay of these two differentials and the properties of this decomposition have several implications on the rational homotopy type of the manifold. In these notes, we review some of these implications. We first study minimal models of the de Rham and Dolbeault algebras for complex manifolds. Then, we present topological properties for compact Kähler manifolds and recall the well-known Formality Theorem of Deligne, Griffiths, Morgan and Sullivan for these spaces.

Another class of topological spaces that carry complex-geometric structures is that of complex algebraic varieties. The underlying complex structure of these spaces manifests in Sullivan’s algebras of forms through the weight and Hodge filtrations. Again, these filtrations and their interplay are extremely powerful in many aspects and in particular, in discovering rational homotopy properties of the varieties. In these notes, we review some of these properties for complex projective varieties. In particular, we will see how in the case of isolated singularities, we can produce formality results.

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**Example 2.1.** The de Rham and Dolbeault cohomology rings of  $\mathbb{C}\mathbb{P}^n$  are given respectively by:

$$H_{dR}^*(\mathbb{C}\mathbb{P}^n) \otimes \mathbb{C} \cong \frac{\mathbb{C}[w_2]}{w_2^{n+1}} \text{ and } H_{\bar{\partial}}^{*,*}(\mathbb{C}\mathbb{P}^n) \cong \frac{\mathbb{C}[w_{1,1}]}{w_{1,1}^{n+1}}$$

where  $w_2$  has degree 2 and  $w_{1,1}$  has bidegree  $(1, 1)$ . For instance, if  $n = 2$  this gives the following Betti and Hodge numbers:

$$\begin{array}{cccc} 1 & & & 1 \\ 0 & & 0 & 0 \\ 1 & & 0 & 1 & 0 \\ 0 & & 0 & 0 \\ 1 & & & 1 \end{array}$$

A Dolbeault model of  $\mathbb{C}\mathbb{P}^n$  is defined as follows. Consider the free graded algebra  $\Lambda(x, y)$  generated by  $x$  and  $y$  in bidegrees  $|x| = (1, 1)$  and  $|y| = (n+1, n)$  respectively. Define a differential  $\bar{\partial}$  of bidegree  $(0, 1)$  by letting  $\bar{\partial}x = y^{n+1}$  and  $\bar{\partial}y = 0$ . A morphism  $(\Lambda(x, y), \bar{\partial}) \rightarrow (\mathcal{A}^{*,*}(M), \bar{\partial})$  is given by sending  $x$  to a representative of  $w_{1,1} \in H_{\bar{\partial}}^{1,1}(\mathbb{C}\mathbb{P}^n)$  and  $y \mapsto 0$ . Note that by forgetting the bidegrees we recover the classical de Rham model for  $\mathbb{C}\mathbb{P}^n$  (this is not always the case!).

**Example 2.2.** Consider the Hopf surface  $\mathcal{M} \cong S^1 \times S^3$ . Its Betti and Hodge numbers are:

$$\begin{array}{cccc} 1 & & & 1 \\ 1 & & 1 & 0 \\ 0 & & 0 & 0 & 0 \\ 1 & & 0 & 1 \\ 1 & & & 1 \end{array}$$

Since  $\mathcal{M}$  is the total space of a holomorphic principal bundle  $\mathbb{T}^1 = S^1 \times S^1 \rightarrow \mathcal{M} \cong S^3 \times S^1 \rightarrow \mathbb{C}\mathbb{P}^1$ , using the Borel spectral sequence, a Dolbeault model can be computed by taking the product of Dolbeault models of the fiber and the base and perturbing the differential by a term of bidegree  $(0, 1)$  (see 4.62 of [FOT08]). A Dolbeault model for  $\mathbb{T}^1$  is given by the free cdga with trivial differential given by  $\Lambda(\alpha, \beta)$  where  $|\alpha| = (0, 1)$ ,  $|\beta| = (1, 0)$ . A Dolbeault model for  $\mathbb{C}\mathbb{P}^1$  is given by  $\Lambda(x, y)$  where  $\bar{\partial}y = x^2$ ,  $|x| = (1, 1)$  and  $|y| = (2, 1)$ . A Dolbeault model for  $\mathcal{M}$  is then given by  $\Lambda(\alpha, \beta, x, y)$  with  $\bar{\partial}\alpha = 0$ ,  $\bar{\partial}\beta = x$ ,  $\bar{\partial}x = 0$  and  $\bar{\partial}y = x^2$ . By forgetting the bidegrees we recover the classical de Rham model for  $S^1 \times S^3$ .

**Example 2.3.** Consider the Calabi-Eckmann manifold  $\mathcal{M}_{1,1} \cong S^3 \times S^3$ . We have:

$$\begin{array}{cccc} 1 & & & 1 \\ 0 & & 1 & 0 \\ 0 & & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & & 0 & 1 & 0 \\ 0 & & 0 & 1 \\ 1 & & & 1 \end{array}$$

A Dolbeault model for  $\mathcal{M}_{3,3}$  is given by  $\Lambda(x, y, z, w)$  with  $|x| = (0, 1)$ ,  $|y| = (1, 1)$ ,  $|z| = |w| = (2, 1)$  and the only non-trivial differential given by  $\bar{\partial}z = y^2$ . To obtain a de Rham model for  $\mathcal{M}_{3,3}$  it suffices to add a differential  $\partial x = y$ .

### 3. FORMALITY OF COMPACT KÄHLER MANIFOLDS

Let  $M$  be a complex manifold and let  $J : TM \rightarrow TM$  be its almost complex structure. A *hermitian metric* on  $M$  is a Riemannian metric  $h$  such that  $h(JX, JY) = h(X, Y)$ .

**Exercise 3.1.** Check that if  $g$  is a Riemannian metric then we may obtain a hermitian metric  $h$  by letting  $h(X, Y) := \frac{1}{2}(g(X, Y) + g(JX, JY))$ . Hence a hermitian metric always exists on an almost complex manifold.

Given a hermitian metric  $h$ , we define its *associated fundamental two-form* by

$$w_h(X, Y) := h(X, JY) \in \mathcal{A}^{1,1}(M).$$

Then  $h$  is said to be a *Kähler metric* if  $dw_h = 0$ . A *Kähler manifold* is a complex manifold admitting a hermitian Kähler metric. Examples of Kähler manifolds:

- The complex projective spaces  $\mathbb{C}\mathbb{P}^n$  (with the Fubini-Study metric).
- Compact Riemann surfaces (since  $dw \in \mathcal{A}^3 = 0$ ).
- Smooth projective varieties.
- The product of two Kähler manifolds is Kähler (Exercise!).
- Any complex submanifold  $N$  of a Kähler manifold  $M$  is Kähler (Exercise!).

The condition of being Kähler imposes strong topological conditions on the manifold. We collect the main properties of its algebra of forms in the following Theorem. The three statements below are all a consequence of Hodge theory and are strongly related.

**Theorem 3.2.** *Let  $M$  be a compact Kähler manifold.*

- (1) ( *$\partial\bar{\partial}$ -lemma*). *If  $\partial x = \bar{\partial}x = 0$  and  $\partial y = x$  then  $x = \partial\bar{\partial}z$  for some  $z$ .*
- (2) *The complex cohomology of  $M$  decomposes as a direct sum*

$$H_{dR}^k(M) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(M) \text{ and } H_{\bar{\partial}}^{p,q}(M) = \overline{H_{\partial}^{q,p}(M)}.$$

- (3) *The Frölicher spectral sequence degenerates at the first stage.*

Note that in particular, for a compact Kähler manifold we have:

$$b^k(M) = h^{k,0}(M) + h^{k-1,1}(M) + h^{k-2,2}(M) + \dots + h^{0,k}(M).$$

Also, the Hodge numbers exhibit special symmetries:  $h^{p,q}(M) = h^{n-p,n-q}(M)$  (duality) and  $h^{p,q}(M) = h^{q,p}(M)$  (complex conjugation).

**Exercise 3.3.** Show that if  $M$  is a compact Kähler manifold of dimension  $n$  then  $b^{2n}(M) \neq 0$  and that odd Betti numbers of  $M$  are either 0 or even.

The Formality Theorem of Deligne, Griffiths, Morgan and Sullivan is a striking application of Hodge theory to the topology of compact Kähler manifolds:

**Theorem 3.4** ([DGMS75]). *Every compact Kähler manifold is a formal topological space.*

*Proof.* By the  $\partial\bar{\partial}$ -lemma there are morphisms of cdga's

$$(A_{dR}(M) \otimes \mathbb{C}, d = \partial + \bar{\partial}) \xleftarrow{j} (\text{Ker}(\partial), \bar{\partial}) \xrightarrow{\pi} (H_{\partial}(M), \bar{\partial})$$

such that  $H(j)$  and  $H(\pi)$  are both injective and surjective (the map  $j$  is just the inclusion, and  $\pi$  is the projection). We leave the details of this claim as an exercise (the solution can be found in [DGMS75]!). Since  $M$  is Kähler we have that  $\bar{\partial}$  is trivial on  $H_{\partial}(M)$  and that  $H_{\bar{\partial}}(M) \cong H_{\partial}(M) \cong H_{dR}(M) \otimes \mathbb{C}$ , so  $(H_{\partial}(M), \bar{\partial}) \cong (H_{dR}(M) \otimes \mathbb{C}, 0)$ . This proves formality over  $\mathbb{C}$ . But then, formality is independent of the base field (see [Sul77], see also [HS79]).  $\square$

The following is an example of a compact complex manifold which is not formal.

**Example 3.5.** The Betti and Hodge numbers for the Iwasawa manifold  $\mathcal{I}$  are:

1				1
4		3		2
8		3	6	2
10	1	6	6	1
8		2	6	3
4			2	3
1				1

One can already see by looking at the Hodge diamond that the Iwasawa manifold is not Kähler. Furthermore, it is not formal. Indeed, a minimal model for  $\mathcal{I}$  is given by  $M = \Lambda(x_1, x_2, y_1, y_2, \theta_1, \theta_2)$  with all generators of degree 1 and the differential given by  $d\theta_1 = x_1y_1 - x_2y_2$  and  $d\theta_2 = x_1y_2 - x_2y_1$ . Assume there exists a morphism  $M \rightarrow H^*(\mathcal{I})$ . Then  $x_i \mapsto [x_i]$ ,  $y_i \mapsto [y_i]$  and  $\theta_i \mapsto 0$ . In degree 6 we get a contradiction.

#### 4. BASICS ON COMPLEX ALGEBRAIC VARIETIES

This section is a very quick and superficial introduction to complex projective varieties. There are uncountable good references to learn complex algebraic geometry. For instance, the book of Griffiths-Harris [GH78] is very accessible.

A *complex projective variety* is given by a subset of  $\mathbb{C}\mathbb{P}^N$  defined by the vanishing of a system of homogeneous polynomial equations over the complex numbers. Every complex projective variety is a compact topological space, with the topology induced by the Euclidean topology of  $\mathbb{C}^n$ . Examples of complex projective varieties:

- The projective space  $\mathbb{C}\mathbb{P}^n$  and its products  $\mathbb{C}\mathbb{P}^n \times \mathbb{C}\mathbb{P}^m$ .
- A compact Riemann surface (i.e., a compact complex manifold of dimension one).
- A compact complex manifold of dimension two with two algebraically independent meromorphic functions.
- Not all Kähler manifolds are projective! The Kodaira embedding gives a criterion for this to happen. Conversely, every smooth projective variety is a compact Kähler manifold.

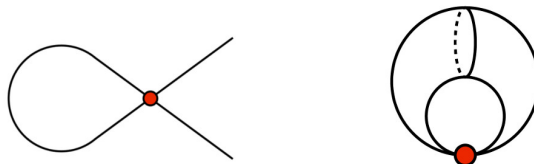
Let  $X$  be a complex projective variety. A point  $x \in X$  is called *regular* if there is an open neighborhood  $U$  of  $x$  in  $\mathbb{C}\mathbb{P}^N$  and homogeneous polynomials  $f_1, \dots, f_m$  in  $N + 1$  variables such that

$$X \cap U = \{(x_0 : \dots : x_N) \in U; f_j(x_0 : \dots : x_N) = 0, 1 \leq j \leq m\}$$

and the Jacobian matrix of partial derivatives  $\frac{\partial f_j}{\partial x_i}$  has rank  $m$  at  $x$ . Otherwise, we say that  $x$  is a *singular point*. A variety is said to be *smooth* if it has no singular points. Denote by  $\Sigma$  the set of singular points of  $X$ . The set  $X_{reg} = X - \Sigma$  of regular points is a dense open subset of  $X$ , which is a complex submanifold of  $\mathbb{C}\mathbb{P}^N$ . The variety  $X$  is said to have *dimension*  $n$  if each connected component of  $X_{reg}$  is a complex manifold of complex dimension  $n$ . Varieties of dimension 1 (resp. 2) are called *curves* (resp. *surfaces*).

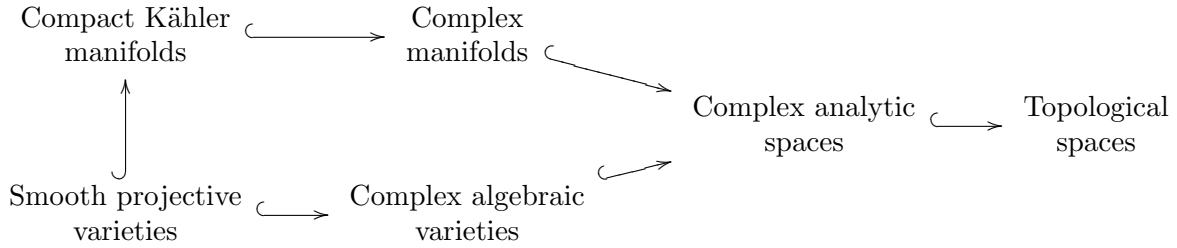
All the examples listed above are smooth varieties. We will see examples of singular varieties in the following pages. For the moment, here is a favorite:

**Example 4.1** (Nodal cubic). Let  $X = \{y^2z - x^2z - x^3 = 0\} \subset \mathbb{C}\mathbb{P}^2$ . This is a projective curve with one singular point  $(0, 0, 1)$ . The nodal cubic is irreducible, meaning that it is not a union of other curves. Topologically, it is just the pinched torus.



Real and complex representations of the nodal cubic

We have the following diagram of strict inclusions:



## 5. THE WEIGHT FILTRATION IN COHOMOLOGY

We next explain the main ideas of the weight filtration in cohomology. For a very first glimpse to mixed Hodge theory, Durfee [Dur83] is very helpful. We also recommend the notes of Griffiths-Schmid [GS75]. The book of Peters-Steenbrink [PS08] and the original works of Deligne [Del71] and [Del74] are more advanced. The lecture notes [GNAPP88] contain the treatment of mixed Hodge theory via cubical hyperresolutions, which may be more accessible than Deligne's theory of simplicial resolutions.

Let  $X$  be a complex projective variety. Deligne introduced a filtration

$$0 = W_{-1}H^k(X) \subset W_0H^k(X) \subset \cdots \subset W_kH^k(X) = H^k(X)$$

on the  $k$ -th rational cohomology of  $X$ , for each  $k \geq 0$ , in such a way that any of the successive quotients

$$Gr_m^W H^k(X) := W_mH^k(X)/W_{m-1}H^k(X)$$

“behaves” as the cohomology of a smooth projective variety, in the sense that it has a Hodge decomposition: the complexification of each of the above quotients admits a direct sum decomposition

$$Gr_m^W H^k(X) \otimes \mathbb{C} \cong \bigoplus_{p+q=m} H^{p,q} \text{ such that } H^{p,q} = \overline{H}^{q,p}.$$

The filtration  $W$  is called the *weight filtration*. Together with the above Hodge decompositions, this data constitutes a *mixed Hodge structure*. If  $X$  is a smooth projective variety, then the weight filtration on  $H^k(X)$  is pure of weight  $k$ , for all  $k \geq n$ , that is,  $0 = W_{k-1}H^k(X) \subset W_kH^k(X) = H^k(X)$ .

Let us briefly explain how  $W$  is defined. The main tool is Hironaka's Theorem on resolution of singularities, which states that for every complex variety  $X$  with singular locus  $\Sigma$ , there exists a cartesian diagram

$$\begin{array}{ccc}
 D & \xrightarrow{j} & \tilde{X} \\
 g \downarrow & & \downarrow f \\
 \Sigma & \xrightarrow{i} & X
 \end{array}$$

where  $\tilde{X}$  is a smooth projective variety,  $f : \tilde{X} \rightarrow X$  is an isomorphism outside  $\Sigma$  and  $D$  is a simple normal crossings divisor (you can think of  $D$  as a codimension 1 subvariety of  $\tilde{X}$  which locally looks like a union of coordinate hyperplanes intersecting transversally). The above square gives a long exact sequence in cohomology

$$\cdots \rightarrow H^k(X) \rightarrow H^k(\Sigma) \oplus H^k(\tilde{X}) \rightarrow H^k(D) \rightarrow H^{k+1}(X) \rightarrow \cdots$$

By iterating this process (i.e., by applying again Hironaka's resolution on  $\Sigma$  and  $D$ , and so on...) one gets a *simplicial resolution*  $X_\bullet \rightarrow X$  of  $X$ : this is a collection of smooth projective varieties  $X_\bullet = \{X_p\}$  together with morphisms

$$\cdots \rightrightarrows X_2 \rightrightarrows X_1 \rightrightarrows X_0 \rightarrow X$$

satisfying cohomological descent (see [Del74], see also [PS08], II.5 for a precise definition or [GNAPP88] for the alternative approach of cubical hyperresolutions). The main idea of cohomological descent is that one may compute the cohomology of  $X$  from the cohomologies of  $X_p$ . The *weight spectral sequence* is given by:

$$E_1^{p,q}(X) := H^q(X_p) \rightrightarrows H^{p+q}(X)$$

where the differential  $d_1 : E_1^{p,q}(X) \rightarrow E_1^{p+1,q}(X)$  is defined via a combinatorial sum of the restriction morphisms of the various maps  $X_p \rightarrow X_{p-1}$ .

Deligne showed that this spectral sequence degenerates at the second stage, and that the induced filtration on the rational cohomology of  $X$  is well-defined (does not depend on the chosen resolution) and is functorial for morphisms of varieties. The *weight filtration*  $W$  on  $H^*(X)$  is defined as the induced filtration. We have  $Gr_q^W H^{p+q}(X) \cong E_2^{p,q}(X)$ . The picture looks like this:

$$E_1^{*,*}(X) = \begin{array}{c} \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ H^4(X_0) \rightarrow H^4(X_1) \rightarrow H^4(X_2) \rightarrow \dots \\ H^3(X_0) \rightarrow H^3(X_1) \rightarrow H^3(X_2) \rightarrow \dots \\ H^2(X_0) \rightarrow H^2(X_1) \rightarrow H^2(X_2) \rightarrow \dots \\ H^1(X_0) \rightarrow H^1(X_1) \rightarrow H^1(X_2) \rightarrow \dots \\ H^0(X_0) \rightarrow H^0(X_1) \rightarrow H^0(X_2) \rightarrow \dots \end{array} \implies E_2^{*,*}(X) \cong \begin{array}{c} \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ Gr_4^W H^4(X) \quad Gr_4^W H^5(X) \quad Gr_4^W H^6(X) \dots \\ Gr_3^W H^3(X) \quad Gr_3^W H^4(X) \quad Gr_3^W H^5(X) \dots \\ Gr_2^W H^2(X) \quad Gr_2^W H^3(X) \quad Gr_2^W H^4(X) \dots \\ Gr_1^W H^1(X) \quad Gr_1^W H^2(X) \quad Gr_1^W H^3(X) \dots \\ Gr_0^W H^0(X) \quad Gr_0^W H^1(X) \quad Gr_0^W H^2(X) \dots \end{array}$$

For readers not familiar with spectral sequences, there is no need to be scared. Indeed, the sentence “degenerates at the second stage” is very good news. What we just have here is a cochain complex

$$E_1^{*,q}(X) : 0 \rightarrow H^q(X_0) \xrightarrow{d^{0,q}} H^q(X_1) \xrightarrow{d^{1,q}} H^q(X_2) \rightarrow \dots$$

for each  $q \geq 0$ . The above picture says that the cohomology of these complexes is

$$H^p(E_1^{*,q}(X), d^{*,q}) := \frac{\text{Ker}(d^{p,q} : E_1^{p,q}(X) \rightarrow E_1^{p+1,q}(X))}{\text{Im}(d^{p-1,q} : E_1^{p-1,q}(X) \rightarrow E_1^{p,q}(X))} \cong Gr_q^W H^{p+q}(X).$$

In particular, if we want to recover the cohomology of  $X$ , we just need to sum over the diagonals of the above right table:

$$H^q(X) \cong Gr_0^W H^q(X) \oplus Gr_1^W H^q(X) \oplus \dots \oplus Gr_q^W H^q(X).$$

We now compute the weight spectral sequence in some particular simple examples.

**Example 5.1.** Let  $X$  be a complex projective variety of dimension  $n$  with only isolated singularities (so that  $\dim \Sigma = 0$ ). Assume that there is a resolution  $f : \tilde{X} \rightarrow X$  such that  $D := f^{-1}(\Sigma)$  is a smooth projective variety of dimension  $n - 1$  (this happens, for instance, when the singularities are ordinary multiple points). Then we have a simplicial resolution  $D \rightrightarrows \Sigma \sqcup \tilde{X} \rightarrow X$  and Deligne’s weight spectral sequence is given by:

$$E_1^{0,q}(X) = H^q(\Sigma) \oplus H^q(\tilde{X}), \quad E_1^{1,q} = H^q(D) \quad \text{and} \quad E_1^{p,q}(X) = 0 \quad \text{for} \quad p > 1.$$

The differential  $d_1 : E_1^{0,q}(X) \rightarrow E_1^{1,q}(X)$  is given by  $d_1(\sigma, x) = j^*(x) - g^*(\sigma)$ , where  $j : D \hookrightarrow \tilde{X}$  is the inclusion and  $g : D \rightarrow \Sigma$  is the restriction of  $f$  to  $D$ .

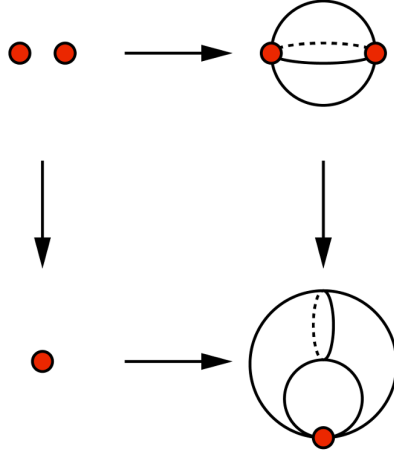
**Subexample 5.2** (Nodal cubic). Let  $C$  be the nodal cubic of Example 4.1. The normalization of  $C$  is  $\mathbb{CP}^1$ . Hence a resolution of  $C$  is given by the following diagram.

By Example 5.1 we have:

$$E_1^{*,*}(C) = \begin{array}{c} \mathbb{Q} \\ 0 \\ \mathbb{Q}^2 \rightarrow \mathbb{Q}^2 \end{array} \implies E_2^{*,*}(C) \cong \begin{array}{c} \mathbb{Q} \\ 0 \\ \mathbb{Q} \quad \mathbb{Q} \end{array}.$$

This gives  $Gr_0^W H^0(C) = \mathbb{Q}$ ,  $Gr_0^W H^1(C) = \mathbb{Q}$  and  $Gr_2^W H^2(C) = \mathbb{Q}$ .

**Example 5.3.** Let  $X$  be a complex projective surface with isolated singularities  $\Sigma$ . Then by Hironaka there exists a resolution  $f : \tilde{X} \rightarrow X$  such that  $D := f^{-1}(\Sigma)$  is a simple normal crossings divisor. We may write  $D = D_1 \cup \dots \cup D_N$  as the union of  $N$  smooth projective curves meeting transversally. Let



Resolution of the nodal cubic

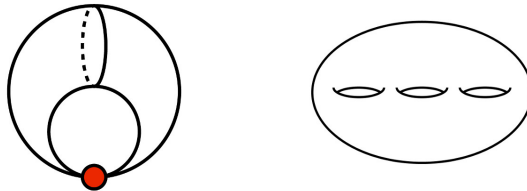
$\tilde{D} := \bigsqcup D_i$  and  $Z := \bigsqcup_{i \neq j} D_i \cap D_j$ . Note that  $\tilde{D}$  is the disjoint union of  $N$  smooth projective curves and  $Z$  is just a finite number of points. Deligne's weight spectral sequence is then given by:

$$E_1^{0,q}(X) = H^q(\tilde{X}) \oplus H^q(\Sigma), \quad E_1^{1,q}(X) = H^q(\tilde{D}) \quad \text{and} \quad E_1^{2,q}(X) = H^q(Z).$$

The differential  $d_1 : E_1^{0,q}(X) \rightarrow E_1^{1,q}(X)$  is given by  $(x, \sigma) \mapsto j^*(x) - g^*(\sigma)$ , where  $j : \tilde{D} \rightarrow \tilde{X}$  and  $g : \tilde{D} \rightarrow \Sigma$  are the obvious maps. Let  $i_1 : Z \rightarrow \tilde{D}$  be given by  $D_i \cap D_j \mapsto D_i$  for  $i < j$  and define  $i_2 : Z \rightarrow \tilde{D}$  by letting  $D_i \cap D_j \mapsto D_j$  for  $i < j$ . The differential  $d_1 : E_1^{1,q}(X) \rightarrow E_1^{2,q}(X)$  is then given by  $i_1^* - i_2^*$ .

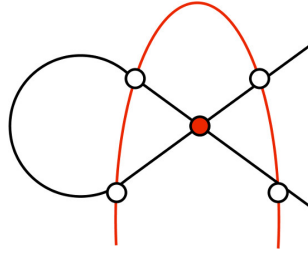
The following is an example of a projective surface with non-trivial weight filtration:

**Subexample 5.4** (Cusp singularity). Let  $C$  be the nodal cubic of Example 4.1, sitting in  $\mathbb{CP}^2$ . We would like to contract  $C$  a point. However, we want to do it so that the contraction is a projective variety. By Castelnuovo's criterion, we are allowed to do this whenever the curve has negative self-intersection. Note that in  $\mathbb{CP}^2$ , we have  $|C \cap C| = 9 > 0$ . To fix this, we choose a smooth quartic  $C'$  in  $\mathbb{CP}^2$  given by a homogeneous polynomial  $f(x, y, z) = 0$  of degree 4 in such a way that it intersects transversally with  $C$  at the smooth points of  $C$ , so that  $|C \cap C'| = 12$ .

The curves  $C$  and  $C'$  in their complex representations

The trick now is to consider the blow-up  $Y = Bl_{12}\mathbb{CP}^2$  of  $\mathbb{CP}^2$  at these 12 points (this procedure substitutes each of the 12 points in  $\mathbb{CP}^2$  by a projective line). We get a space  $Y$  that is homeomorphic to the connected sum of 13 projective planes  $\mathbb{CP}^2$ . The proper transform  $\tilde{C} \cong C$  of  $C$  in  $Y$  satisfies





Real representation of the curves  $C$  and  $C'$  intersecting in general position

$|\tilde{C} \cap \tilde{C}'| = -3 < 0$ . Hence now we may contract  $\tilde{C}$  to a point, and get a normal projective surface  $X$ :

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & X \end{array}$$

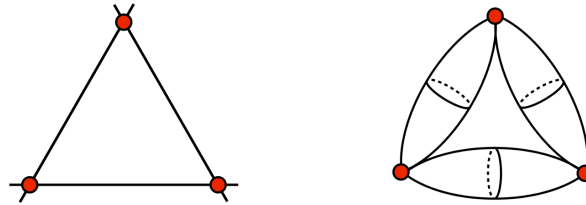
The variety  $X$  can be described as the set of points  $(x, y, z, w) \in \mathbb{C}\mathbb{P}^3$  such that:

$$X = \{w(y^2z - x^2z - x^3) + f(x, y, z) = 0\} \subset \mathbb{C}\mathbb{P}^3.$$

To find a simplicial resolution of  $X$  we turn  $\tilde{C}$  into a simple normal crossings divisor. This can be done by blowing-up  $Y$  twice the singular point of  $\tilde{C}$ . We get

$$\begin{array}{ccc} D & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & X \end{array}$$

where  $D$  is a cycle of three projective lines, as shown in the picture below and  $\tilde{X}$  is homeomorphic to the connected sum of 15 projective planes.



Real and complex representations of the divisor  $D$

Now we may apply Example 5.3. The variety  $\tilde{D}$  is the disjoint union of 3 projective lines, while  $Z$  is given by 3 points. We get:

$$E_1^{*,*}(X) = \begin{array}{|c} \mathbb{Q} \\ 0 \\ \mathbb{Q}^{15} \twoheadrightarrow \mathbb{Q}^3 \\ 0 \quad 0 \\ \mathbb{Q}^2 \longrightarrow \mathbb{Q}^3 \longrightarrow \mathbb{Q}^3 \end{array} \implies E_2^{*,*}(X) \cong \begin{array}{|c} \mathbb{Q} \\ 0 \\ \mathbb{Q}^{12} \quad 0 \\ 0 \quad 0 \\ \mathbb{Q} \quad 0 \quad \mathbb{Q} \end{array} .$$

Hence  $H^2(X)$  has a non-trivial weight filtration:

$$Gr_0^W H^2(X) \cong \mathbb{Q}, Gr_1^W H^2(X) = 0 \text{ and } Gr_2^W H^2(X) \cong \mathbb{Q}^{12}.$$

## 6. THE WEIGHT FILTRATION IN RATIONAL HOMOTOPY

For the applications to rational homotopy we are interested in a multiplicative version of Deligne's weight spectral sequence. This was first defined by Morgan [Mor78] in the case of smooth quasi-projective varieties and by Hain [Hai87] and Navarro-Aznar [NA87] independently, for general algebraic varieties. A description of the multiplicative weight spectral sequence in terms of a resolution of singularities, in the case of projective varieties with isolated singularities is given in [CC].

**Theorem 6.1** ([Mor78],[NA87], [Hai87]). *The Sullivan algebra  $\mathcal{A}_{pl}(X)$  of every complex algebraic variety  $X$  carries functorial mixed Hodge structures. In particular, the weight filtration  $W$  on  $\mathcal{A}_{pl}(X)$  defines a multiplicative weight spectral sequence  $(\hat{E}_1(X), d_1)$  which is quasi-isomorphic to Deligne's weight spectral sequence  $(E_1(X), d_1)$  as a cochain complex.*

The multiplicative weight spectral sequence  $(\hat{E}_1(X), d_1)$  is a well-defined algebraic invariant of  $X$  in the homotopy category of differential bigraded algebras. The main advantage of this object with respect to Deligne's weight spectral sequence is that the former carries information about the rational homotopy type of  $X$ :

**Theorem 6.2** ([CG14]). *Let  $X$  be a complex algebraic variety. There is a string of quasi-isomorphisms of complex cdga's from  $(\mathcal{A}(X) \otimes \mathbb{C}, d)$  to  $(\hat{E}_1(X) \otimes \mathbb{C}, d_1)$  compatible with  $W$ .*

*Proof.* The proof uses Sullivan's theory of minimal models together with the Hodge decompositions of a mixed Hodge structure. The main idea is to use Theorem 6.1 to construct a Sullivan minimal model  $M \rightarrow \mathcal{A}(X)$  of the algebra of forms of  $X$  in such a way that for each degree  $n \geq 0$ ,  $M^n$  carries a mixed Hodge structure, with products and differentials compatible with such structures. The second step is to show that, over  $\mathbb{C}$ , the algebra  $M$  admits a splitting, so that  $M \otimes \mathbb{C} \cong \hat{E}_1(M) \otimes \mathbb{C}$ , where  $\hat{E}_1(M)$  is defined via the weight filtration of  $M$ . inner weight filtration. We refer to [CG14] for details.  $\square$

In particular, a Sullivan model for a complex algebraic variety  $X$ , can be computed from  $(\hat{E}_1(X), d_1)$ . We next explain how to compute this differential bigraded algebra in terms of a resolution of singularities, in two very particular examples. For more general constructions we call upon the imagination of the reader and perhaps some knowledge on Navarro's definition of the Thom-Whitney simple functor [NA87]. Alternatively, see [Mor78] for the construction in the case of smooth quasi-projective varieties and [CC] for the case of projective varieties with isolated singularities.

**Example 6.3.** In the situation of Example 5.1, the multiplicative weight spectral sequence for  $X$  is:

$$\begin{array}{ccc} \hat{E}_1^{0,q}(X) & \longrightarrow & H^q(D) \otimes \Lambda(t) \quad , \\ \downarrow & \lrcorner & \downarrow (\delta^0, \delta^1) \\ H^q(\Sigma) \times H^q(\tilde{X}) & \xrightarrow{\begin{pmatrix} g^* & 0 \\ 0 & j^* \end{pmatrix}} & H^q(D) \times H^q(D) \end{array}$$

$\hat{E}_1^{1,q}(X) = H^q(D) \otimes \Lambda(t) \otimes dt$  and  $\hat{E}_1^{p,q}(X) = 0$  for  $p > 1$ . The map  $d_1 : \hat{E}_1^{0,q}(X) \rightarrow \hat{E}_1^{1,q}(X)$  is defined by differentiation with respect to  $t$ . The non-trivial products of  $\hat{E}_1^{*,*}(X)$  are the maps  $\hat{E}_1^{0,q}(X) \times \hat{E}_1^{0,q'}(X) \rightarrow \hat{E}_1^{0,q+q'}(X)$  given by  $(x, a(t)) \cdot (y, b(t)) = (x \cdot y, a(t) \cdot b(t))$  and the maps  $\hat{E}_1^{0,q}(X) \times \hat{E}_1^{1,q'}(X) \rightarrow \hat{E}_1^{1,q+q'}(X)$  given by  $(x, a(t)) \cdot b(t)dt = a(t) \cdot b(t)dt$ .

**Example 6.4.** Let  $X$  be a complex projective surface with isolated singularities. With the notation of Example 5.3, we have:

$$\begin{array}{ccc} \hat{E}_1^{0,q}(X) & \longrightarrow & H^q(\tilde{D}) \otimes \Lambda(t) \quad , \\ \downarrow & \lrcorner & \downarrow (\delta^0, \delta^1) \\ H^q(\Sigma) \times H^q(\tilde{X}) & \xrightarrow{\begin{pmatrix} g^* & 0 \\ 0 & j^* \end{pmatrix}} & H^q(\tilde{D}) \times H^q(\tilde{D}) \end{array}$$

$\hat{E}_1^{1,q}(X) = H^q(Z) \otimes \Lambda(t) \oplus H^q(\tilde{D}) \otimes \Lambda(t) \otimes dt$  and  $\hat{E}_1^{2,q}(X) = H^q(Z) \otimes \Lambda(t) \otimes dt$ . The differentials are defined via the restriction maps of Example 5.3 together with differentiation with respect to  $t$ . The products are defined as in Example 6.3.

We have seen that the complex homotopy of a complex algebraic variety can be computed using the multiplicative weight spectral sequence. On the other hand, we have seen how one may write this spectral sequence in terms of cohomologies of smooth projective varieties. We next use these results facts to prove formality for a large family of complex projective varieties. Most of the results of this section are proven in [CC].

The following first result can be thought of an abstract version of the Formality Theorem of [DGMS75] in the setting of complex algebraic geometry.

**Theorem 6.5** (Purity implies formality). *Let  $X$  be a complex algebraic variety. Assume that the weight filtration on  $H^k(X)$  is pure of weight  $k$  for all  $k \geq 0$ . Then  $X$  is formal.*

*Proof.* It is an easy exercise using Theorem 6.2. (Hint: it suffices to define an injection  $(E_2(X), 0) \cong (\text{Ker}(d_1), 0) \hookrightarrow (\hat{E}_1(X), d_1)$  and to check that this is compatible with products. This gives formality over  $\mathbb{C}$ . Then apply independence of formality on the base field to get formality over  $\mathbb{Q}$ .)  $\square$

**Example 6.6.** Let  $X$  be a complex projective variety whose rational cohomology satisfies Poincaré duality. Then  $X$  is formal.

**Example 6.7.** Let  $X$  be a complex projective variety of dimension  $n$ . Assume that  $X$  is a  $\mathbb{Q}$ -homology manifold (for all  $x \in X$ ,  $H_{\{x\}}^k(X) = 0$  for  $k \neq 2n$  and  $H_{\{x\}}^{2n}(X) \cong \mathbb{Q}$ ). Then  $X$  is formal.

By purely topological reasons we know that every simply connected, 4-dimensional CW-complex is formal. We also know there exist non-formal 4-dimensional CW-complexes. Thanks to deep results of Simpson and Kapovich-Kollár [KK14] we know that there exist non-formal complex projective surfaces. However, if the singularities are normal (i.e., the link of every singular point is connected), we have:

**Theorem 6.8** ([CC]). *Every normal complex projective surface is formal.*

*Proof.* The multiplicative weight spectral sequence for a normal surface has the form

$$\hat{E}_1^{*,*}(X) = \begin{array}{c} E^{0,4} \\ E^{0,3} \\ E^{0,2} \xrightarrow{d^{0,2}} E^{1,2} \\ E^{0,1} \xrightarrow{d^{0,1}} E^{1,1} \\ E^{0,0} \xrightarrow{d^{0,0}} E^{1,0} \xrightarrow{d^{1,0}} E^{2,0} \end{array} \implies E_2^{*,*}(X) \cong \begin{array}{ccc} H^4(X) & & \\ H^3(X) & & \\ \text{Ker}(d^{0,2}) & 0 & \\ \text{Ker}(d^{0,1}) & \text{Coker}(d^{0,1}) & \\ \text{Ker}(d^{0,0}) & 0 & \text{Coker}(d^{1,0}) \end{array}$$

By Theorem 6.2 it suffices to define a quasi-isomorphism from  $(E_2^{*,*}(X), 0)$  to  $(\hat{E}_1^{*,*}(X), d_1)$ . This is done by taking a section  $E_2^{*,*}(X) \rightarrow \text{Ker}(d^{*,*}) \subset \hat{E}_1^{*,*}(X)$  of the projection  $\text{Ker}(d^{*,*}) \rightarrow E_2^{*,*}(X)$ . Then one needs to prove that this gives a morphism compatible with products. Note that there are only a couple of verifications to do. The details can be found in [CC].  $\square$

**Example 6.9** (Cusp singularity). Consider the singular surface  $X$  defined in Subexample 5.4. Since  $X$  is simply connected, we may compute the rational homotopy groups of  $X$  with their weight filtration from a bigraded minimal model  $\rho : M \xrightarrow{\sim} E_2^{*,*}(X)$  of the bigraded algebra  $E_2^{*,*}(X)$ . The weight filtration on  $\pi_i := \pi_i(X) \otimes \mathbb{Q}$  satisfies  $Gr_q^W \pi_{p+q} \cong \text{Hom}(Q(M)^{p,q}, \mathbb{Q})$ , where  $Q(M)^{p,q}$  denotes the indecomposables of  $M$  of bidegree  $(p, q)$ . We may write  $E_2^{*,*}(X) \cong \mathbb{Q}[\alpha, \gamma_1, \dots, \gamma_{12}]$  where the generators have bidegree  $|\alpha| = (2, 0)$  and  $|\gamma_i| = (0, 2)$ . The only non-trivial products are given by  $\gamma_i^2 = -T$  and  $\gamma_i \cdot \gamma_j = T$ , for all  $i \neq j$ . We compute the first steps of a minimal model for  $E_2^{*,*}(X)$ . Let  $M_2$  be the free bigraded algebra  $M_2 = \Lambda(\bar{\alpha}, \bar{\gamma}_1, \dots, \bar{\gamma}_{12})$  with trivial differential generated by

elements of bidegree  $|\bar{\alpha}| = (2, 0)$  and  $|\bar{\gamma}_i| = (0, 2)$ . Then the map  $\rho_2 : M_2 \rightarrow E_2^{*,*}(X)$  given by  $\bar{x} \mapsto x$  is a 2-quasi-isomorphism of bigraded algebras. Hence we have

$$Gr_0^W \pi_2 \cong \mathbb{Q}, Gr_1^W \pi_2 = 0 \text{ and } Gr_2^W \pi_2 \cong \mathbb{Q}^{12}.$$

Let  $M_3 = M_2 \otimes_d \Lambda(V_{3,0}, V_{-1,4})$  where  $V_{i,j}$  are the graded vector spaces of pure bidegree  $(i, j)$  given by  $V_{3,0} = \mathbb{Q}\langle x \rangle$  and  $V_{-1,4} = \mathbb{Q}\langle \xi_{ij} \rangle$ , with  $1 \leq i, j \leq 12$  and  $(i, j) \neq (1, 1)$ . The differentials are given by  $dx = \bar{\alpha}^2$  and  $d\xi_{ij} = \bar{\gamma}_i \bar{\gamma}_j$ . Then the extension  $\rho_3 : M_3 \rightarrow E_2^{*,*}(X)$  of  $\rho_2$  given by  $V_{i,j} \mapsto 0$  is a 3-quasi-isomorphism. The formula  $Gr_p^W \pi_3 \cong \text{Hom}(V_{3-p,p}, \mathbb{Q})$  gives:

$$Gr_0^W \pi_3 \cong \mathbb{Q}, Gr_1^W \pi_3 = 0, Gr_2^W \pi_3 \cong \mathbb{Q}^{12}, Gr_3^W \pi_3 = 0 \text{ and } Gr_4^W \pi_3 \cong \mathbb{Q}^{77}.$$

We invite the reader to compute some next steps.

Theorem 6.8 generalizes to projective varieties of arbitrary dimension as follows:

**Theorem 6.10** ([CC]). *Let  $X$  be a complex projective variety of dimension  $n$  with normal isolated singularities. Denote by  $\Sigma$  the singular locus of  $X$ , and for each  $\sigma \in \Sigma$  let  $L_\sigma$  denote the link of  $\sigma$  in  $X$ . If  $\tilde{H}^k(L_\sigma) = 0$  for all  $k \leq n - 2$  for every  $\sigma \in \Sigma$ , then  $X$  is a formal topological space.*

*Proof.* The proof is a straightforward generalization of that of Theorem 6.8 after showing that, thanks to the conditions on the link, the weight spectral sequence for  $X$  has the form

$$\hat{E}_1^{*,*}(X) = \begin{array}{c} 2n \\ \bullet \\ \bullet \\ \bullet \bullet \\ \bullet \bullet \\ n \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \\ 0 \\ \bullet \bullet \bullet \bullet \bullet \\ n \end{array} \implies E_2^{*,*}(X) = \begin{array}{c} 2n \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ n \\ \bullet \bullet \\ \bullet \bullet \\ \bullet \bullet \\ 0 \\ \bullet \bullet \bullet \bullet \\ n \end{array}$$

where the bullets denote the non-trivial elements. See [CC] for details. □

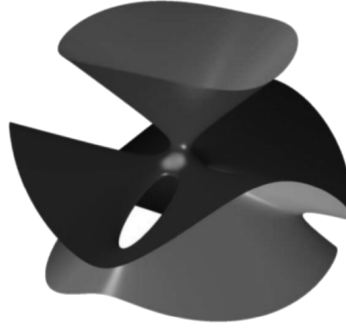
**Example 6.11.** The above theorem gives formality for the following spaces:

- Hypersurfaces with isolated singularities.
- Complete intersections with isolated singularities.
- Projective varieties whose singularities are ordinary multiple points.
- Projective cones over smooth projective varieties.

**Example 6.12.** The Segre cubic  $S$  is a simply connected projective threefold with 10 ordinary singular points, and is described by the set of points  $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$  of  $\mathbb{CP}^5$

$$S : \{x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\}.$$

A resolution  $f : \overline{\mathcal{M}}_{0,6} \rightarrow S$  of  $S$  is given by the moduli space  $\overline{\mathcal{M}}_{0,6}$  of stable rational curves with 6 marked points, and  $f^{-1}(\Sigma) = \bigsqcup_{i=1}^{10} \mathbb{CP}^1 \times \mathbb{CP}^1$ , where  $\Sigma = \{\sigma_1, \dots, \sigma_{10}\}$  is the singular locus of  $S$ .



Real representation of the Segre cubic

By Example 5.1 we have

$$E_1^{*,*}(S) \cong \begin{array}{|c|} \hline \mathbb{Q} \\ \hline 0 \\ \hline \mathbb{Q}^{16} \rightarrow \mathbb{Q}^{10} \\ \hline 0 \quad 0 \\ \hline \mathbb{Q}^{16} \rightarrow \mathbb{Q}^{20} \\ \hline 0 \quad 0 \\ \hline \mathbb{Q}^{11} \rightarrow \mathbb{Q}^{10} \\ \hline \end{array} \implies E_2^{*,*}(S) = \begin{array}{|c|} \hline \mathbb{Q} \\ \hline 0 \\ \hline \mathbb{Q}^6 \quad 0 \\ \hline 0 \quad 0 \\ \hline \mathbb{Q} \quad \mathbb{Q}^5 \\ \hline 0 \quad 0 \\ \hline \mathbb{Q} \quad 0 \\ \hline \end{array} .$$

Hence  $S$  has a non-trivial weight filtration, with  $0 \neq Gr_2^W H^3(S) \cong \mathbb{Q}^5$ . Since  $S$  is simply connected, we may compute the rational homotopy groups  $\pi_*(S) \otimes \mathbb{Q}$  with their weight filtration from a minimal model of  $E_2^{*,*}(S) \cong \mathbb{Q}[a, b_1, \dots, b_5, c_0, \dots, c_5, e]$  with the only non-trivial products  $a^2 = c_0$  and  $a^3 = e$ . The bidegrees are given by  $|a| = (0, 2)$ ,  $|b_i| = (1, 2)$ ,  $|c_i| = (0, 4)$  and  $e = (0, 6)$ . In low degrees:

$$\begin{aligned} Gr_2^W \pi_2 &\cong \mathbb{Q}, Gr_2^W \pi_3 \cong \mathbb{Q}^5, Gr_4^W \pi_4 \cong \mathbb{Q}^5, Gr_3^W \pi_5 \cong \mathbb{Q}^{10}, Gr_5^W \pi_5 \cong \mathbb{Q}^5, \\ Gr_4^W \pi_6 &\cong \mathbb{Q}^{25}, Gr_5^W \pi_6 \cong \mathbb{Q}^{25}, Gr_4^W \pi_7 \cong \mathbb{Q}^{40}, Gr_5^W \pi_7 \cong \mathbb{Q}^{50}, Gr_7^W \pi_7 \cong \mathbb{Q}^{26}. \end{aligned}$$

We leave as an exercise to compute the higher degrees.

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