ON ALGEBRAIC ASPECTS OF THE MODULI SPACE OF FLAT CONNECTIONS

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1. Overview

These notes are based on two seminar talks given in a seminar on moduli of flat connections organised by Ivan Contreras at the University of Zürich, 2013. The main aim of the seminar was to describe, from a gauge theoretic point of view, the moduli space of flat connections on a principal G-bundle over a compact Riemann surface X, for a compact and connected Lie group G. A secondary aim was to relate flat connections on a principal G-bundle over X to G-representations of the fundamental group of X and algebraic G-bundles over X. These notes focus mostly on the algebraic aspects and, in particular, we study the case of $G = \mathrm{U}(n)$. In these notes, we treat the following topics.

- The Atiyah–Bott isomorphism between the space of holomorphic structures on a principal bundle $P_{\mathbb{C}}$ for the complexified group $G_{\mathbb{C}}$ over X and G-connections on a fixed reduction $P \subset P_{\mathbb{C}}$ to $G \subset G_{\mathbb{C}}$.
- For $G = \mathrm{U}(n)$, we describe the Narasimhan–Seshadri correspondence that relates projectively flat unitary connections on a Hermitian vector bundle over X with polystable holomorphic (or, equivalently, algebraic) vector bundle structures over X.
- The construction of the moduli space of stable algebraic vector bundles on a smooth complex projective algebraic curve X via geometric invariant theory.
- For $G = \mathrm{U}(n)$, the non-abelian Hodge correspondence that relates three different moduli spaces (the so-called de Rham, Betti and Dolbeault moduli spaces).

The main aim of these notes is to explain the key ideas to a diverse audience succinctly and, at the same time, to assume as little background as possible. Although these notes focus on algebraic aspects, they are written in order to minimise the algebro-geometric prerequisites and make the notes more accessible to a general audience (the background of most of the participants of the seminar was in mathematical physics rather than algebraic geometry). Due to the limited amount of time for the two seminars and the breadth of these topics, our focus is on giving an overall outline of the main ideas and so, for more technical aspects that we omit, we provide references. There are many excellent references that cover these topics in much greater and broader depth and we try to give suitable references to the literature when possible; however, the list of references contained in these notes does not touch the vast quantity of work in this area and it is indeed probable that some important references have been forgotten due to my own ignorance or forgetfulness.

The notes are essentially split into two parts: §§2-4 contain the necessary background material and §§5-8 contain the contents of the two seminars on the algebraic aspects. The content of the background sections is as follows. In §2, we give the definition of a connection on a principal G-bundle in terms of an equivariant horizontal distribution, a connection 1-form and local 1-forms satisfying a compatibility condition. In §3, we describe the set of connections on a principal G-bundle as an infinite dimensional symplectic affine space. The gauge group, consisting of G-bundle automorphisms, naturally acts by pulling back connections and we construct the 'de Rham' moduli space of (projectively) flat connections on a principal G-bundle as a symplectic reduction. In §4, we describe the Riemann–Hilbert correspondence relating flat connections on a principal G-bundle over X to representations of the fundamental group of X to G. We also construct the 'Betti' moduli space of (twisted) representations of the fundamental group of X to G.

The content of the two seminar talks is contained in §§5-8. In §5, for a compact group G with complexification $G_{\mathbb{C}}$, we describe the Atiyah–Bott isomorphism between the space of holomorphic structures on a $G_{\mathbb{C}}$-principal bundle over X and G-connections on a reduction to $G \subset G_{\mathbb{C}}$. We also state, without proof, the Hitchin-Kobayashi correspondence that relates Hermitian–Einstein connections (these are $G_{\mathbb{C}}$-connections that are projectively flat and extended from a reduction to G) to polystable holomorphic $G_{\mathbb{C}}$-bundle structures. We then restrict to the case when $G = \mathrm{U}(n)$ and view principal G-bundles as rank n Hermitian vector bundles by taking the vector bundle associated to the standard representation of $\mathrm{U}(n)$ on $\mathbb{C}^n$. In §6, we describe Donaldson's proof of the Narasimhan–Seshadri correspondence that relates projectively flat unitary connections to polystable holomorphic vector bundles. In this section, we give the definitions of stability, semistability and polystability for holomorphic vector bundles and describe certain properties of these bundles that will be needed later on. In §7, we describe the construction of the 'Dolbeault' moduli space of (poly)stable algebraic vector bundles on X as an algebraic variety. We start by giving the definitions of coarse and fine moduli spaces and then explain how geometric invariant theory can be used to constructed moduli spaces as categorical quotients of group actions. After a quick tour of geometric invariant theory, we move on to give the construction of the Dolbeault moduli space. Finally, in §8, we state the non-abelian Hodge theorem: there is a real analytic isomorphism between the de Rham and Betti moduli spaces and a homeomorphism between the Betti and Dolbeault moduli spaces. In particular, we describe in detail the three moduli space for the group $G = \mathrm{U}(1)$.

2. Connections on principal G-bundles

In this section, we give the basic definitions concerning connections on principal bundles. The two books by Kobayashi and Nomizu [7, 8] provide an excellent reference for this material as well as some aspects discussed in later sections.

Let $\pi : P \to X$ be a smooth principal G-bundle i.e. a fibre bundle with a smooth fibre-preserving right G-action such that the action on fibres is free and transitive. We recall the following standard result:

Lemma 2.1. *A principal G-bundle $\pi : P \to X$ is trivial (that is, $P \cong X \times G$ as G-bundles) if and only if $\pi : P \to X$ admits a global section.*

Proof. Given a global section $s : X \to P$, we define $\varphi : X \times G \to P$ by $\varphi(x, g) = s(x) \cdot g$. It is easy to verify that φ is equivariant and preserves fibres, thus φ is a map of principal G-bundles over X. Any map of principal G-bundles is an isomorphism and so P is trivial.

Conversely, given a G-bundle isomorphism $\varphi : X \times G \cong P$, we define a global section $s : X \to P$ by $s(x) = \varphi(x, e)$. □

A principal G-bundle locally admits a section, as locally it is trivialisable. Therefore, if we choose a cover $\{U_\alpha\}$ of X on which P is trivial, then there are local sections $s_\alpha : U_\alpha \to P$. The transition functions for this trivialisation are maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, such that $s_\beta = s_\alpha \cdot g_{\alpha\beta}$.

Let $p \in P$ and $x = \pi(p)$. The vertical tangent space at $p \in P$ is $T_p^v P := \mathrm{Ker}(d_p\pi : T_pP \to T_xX)$ or, equivalently, $T_p^v P = T_p(P_x)$. We have a short exact sequence of vector bundles over P

$$0 \to T^v P \to TP \to \pi^* TX \to 0 \tag{1}$$

The action $P \times G \to P$ has an associated infinitesimal action $\rho : P \times \mathfrak{g} \to TP$

$$(p, \eta) \mapsto \rho_p(\eta) = \frac{d}{dt} p \cdot \exp(t\eta)|_{t=0} \in T_pP.$$

The infinitesimal action $\rho_p : \mathfrak{g} \to T_pP$ is an isomorphism onto the vertical tangent space $T_p^v P$, since the G-action on P is free and we check that

$$d_p\pi(\rho_p(\eta)) = \frac{d}{dt}\pi(p \cdot \exp(t\eta))|_{t=0} = \frac{d}{dt}\pi(p)|_{t=0} = 0.$$

Definition 2.2. A (principal or G-)connection A on P is a G-equivariant horizontal distribution; that is, a smooth splitting of the short exact sequence

$$0 \to T^vP \to TP \to \pi^*TX \to 0$$

which is equivariant. So, for $p \in P$, we have splittings $T_pP = T_p^vP \oplus T_p^hP$ such that $d_p\pi|_{T_p^hP} : T_p^hP \cong T_{\pi(p)}X$ and equivariance means that $T_{pg}^hP = R_{g*}(T_p^hP)$ where $R_g : P \to P$ denotes right multiplication by g.

A connection 1-form is a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P) \otimes \mathfrak{g}$ such that:

(1) A is G-invariant (i.e. $R_g^*A = \mathrm{Ad}_{g^{-1}}A$);

(2) A restricts to the identity on the vertical tangent space: $A_p \circ \rho_p = \mathrm{id}_{\mathfrak{g}}$.

These notions are equivalent as we define $T_p^hP := \mathrm{Ker}A_p$.

Example 2.3. Consider the trivial principal G-bundle $\pi : G \to \mathrm{pt}$; the infinitesimal action is

$$B_g = \frac{d}{dt}(g \cdot \exp(tB))|_{t=0} = g \cdot B,$$

for $B \in \mathfrak{g}$ and $g \in G$. There is a canonical connection θ on $\pi : G \to \mathrm{pt}$ defined by

$$\theta_g = L_{g^{-1}*} : T_gG \to T_eG \cong \mathfrak{g}$$

which we check satisfies the defining properties of a connection:

$$\begin{aligned} &1) \quad \theta_g(B_g) = L_{g^{-1}*}(B_g) = \frac{d}{dt}(g^{-1} \cdot g \cdot \exp(tB))|_{t=0} = B, \\ &2) \quad R_h^*\theta_g(X) = \theta_{gh}(Xh) = L_{(gh)^{-1}*}(Xh) = L_{h^{-1}}\theta_g(Xh) = \mathrm{Ad}_{h^{-1}}\theta_g(X). \end{aligned}$$

This 1-form $\theta \in \Omega^1(G) \otimes \mathfrak{g}$ is called the Maurer–Cartan form of G and satisfies a structural equation (also known as the Maurer–Cartan equation):

$$d\theta + \frac{1}{2}[\theta, \theta] = 0$$

where $[\theta,\theta](X,Y) := [\theta(X),\theta(Y)] - [\theta(Y),\theta(X)] = 2[\theta(X),\theta(Y)]$. To prove this structural equation we take tangent vector $v, w \in T_gG$ and extend to left invariant vector fields V, W on G. Then $\theta(V)$ and $\theta(W)$ are constant and so we have

$$d\theta_g(v,w) = d\theta_g(V,W) = V(\theta(W))_g - W(\theta(V))_g - \theta_g([V,W]) = -\theta_g([V,W])$$

and

$$\theta_g([V,W]) = [\theta_g(V), \theta_g(W)] = [\theta_g(v), \theta_g(w)] = \frac{1}{2}[\theta,\theta]_g(v,w).$$

The structural equation corresponds to the fact that the Maurer–Cartan form is flat.

Remark 2.4. For a connection A on a principal G-bundle P over X, fibrewise the connection A satisfies the structural equation $dA + \frac{1}{2}[A, A] = 0$ (as each fibre is the trivial principal G-bundle $\pi : G \to \mathrm{pt}$). However, globally this fails and so we construct a curvature 2-form for a connection which can be seen as an obstruction to a connection being flat; see §2.3 below for more details.

Let $\{U_\alpha\}$ be an open cover of X on which P is trivial; i.e. we have local sections $s_\alpha : U_\alpha \to P$. If the transition functions are given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, then $s_\beta = s_\alpha \cdot g_{\alpha\beta}$. For a connection 1-form $A \in \Omega^1(P) \otimes \mathfrak{g}$, we define local 1-forms $A_\alpha = s_\alpha^*A \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$. These do not fit together to define a global 1-form on X, but we have the following transformation rule for the overlaps $U_\alpha \cap U_\beta$:

$$A_\beta = \mathrm{ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^*\theta \tag{2}$$

where $\theta \in \Omega^1(G) \otimes \mathfrak{g}$ is the Maurer-Cartan form. A collection of local 1-forms $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ defined on an open cover of X on which P is trivial that satisfy the transformation rule (2) define a connection $A \in \Omega^1(P) \otimes \mathfrak{g}$ (for example, see [7], Chapter I, Proposition 1.4).

Hence there are three equivalent ways to define a connection on P:

(1) as an equivariant horizontal distribution T^hP;

(2) as a G-invariant $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P) \otimes \mathfrak{g}$ that restricts to the identity on the vertical tangent space;
(3) as $\mathfrak{g}$-valued local 1-forms $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ on a cover $\{U_\alpha\}$ of X over which P is trivial that satisfy the transformation rule (2).

2.1. **Connections on vector bundles.** Given a representation $\rho : G \to \mathrm{GL}(V)$, we can construct the associated vector bundle $P(V) = P \times_G V$ as the quotient of $P \times V$ by the G-action

$$g \cdot (p, g) = (p \cdot g, \rho(g^{-1})v).$$

This is a fibre bundle with fibre V. If we take a cover $\{U_\alpha\}$ of X over which P is trivial and has transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, then $P(V)$ is trivial on this cover too with transition functions $h_{\alpha\beta} := \rho \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$.

Example 2.5. (Atiyah sequence of a principal bundle). Let $\pi : P \to X$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G on its Lie algebra $\mathfrak{g}$; then we construct the associated vector bundle $\mathrm{ad}P = P \times_G \mathfrak{g}$ with fibre $\mathfrak{g}$. As $\mathrm{ad}P = (P \times \mathfrak{g})/G$, its pullback along $\pi : P \to X$ can naturally be identified with the vertical tangent bundle $T^vP \cong P \times \mathfrak{g}$. The short exact sequence (1) of vector bundles over P can then be seen as a pullback of the short exact sequence on X (the Atiyah sequence of P)

$$0 \to \mathrm{ad}P \to \pi_*^G TP \to TX \to 0 \tag{3}$$

where $\pi_*^G TP$ denotes the G-invariant direct image i.e. $(\pi_*^G TP)_x = \Gamma(TP|_{\pi^{-1}(x)})^G$. This gives a fourth description of a connection on P: a G-equivariant splitting of (1) is equivalent to a splitting of the Atiyah sequence (3).

Definition 2.6. For a vector bundle $E \to X$, we define an E-valued k-form on X to be a vector bundle map $\wedge^k TX \to E$; that is, a section of $\wedge^k T^*X \otimes E$. We write

$$\Omega^k(X, E) := \Gamma(\wedge^k T^*X \otimes E)$$

and for the trivial bundle $X \times V$, we write

$$\Omega^k(X; V) := \Gamma(\wedge^k T^*X \otimes (X \times V)) = \Omega^k(X) \otimes V.$$

We list some operations on differential forms which can be extended to vector valued forms.

(1) (Pullback) Let $f : Y \to X$; then we define $f^* : \Omega^k(X, E) \to \Omega^k(Y, f^*E)$ by

$$(f^*\alpha)_y(v_1, \dots, v_k) := \alpha_f(y)(d_yf(v_1), \dots, d_yf(v_k)) \in E_{f(y)} \cong (f^*E)_y.$$

(2) (Wedge) We define $- \wedge - : \Omega^k(X, E) \to \Omega^l(X, F) \to \Omega^{k+l}(X, E \otimes F)$ by

$$(\alpha \wedge \beta)_x(v_1, \dots, v_{k+1}) = \frac{1}{l!k!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\alpha_x(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \otimes \eta_x(v_{\sigma(k+1)}, \dots v_{\sigma(k+l)}).$$

(3) (Pairing) For a pairing $\kappa : V \otimes W \to U$ of vector spaces U,V and W, we define the pairing $\kappa(- \wedge -) : \Omega^k(X; V) \otimes \Omega^l(X; W) \to \Omega^{k+l}(X, U)$ by

$$\kappa(\alpha \wedge \beta)_x(v_1, \dots, v_{k+1}) = \frac{1}{l!k!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\kappa\left(\alpha_x(v_{\sigma(1)}, \dots, v_{\sigma(k)}), \eta_x(v_{\sigma(k+1)}, \dots v_{\sigma(k+l)})\right).$$

(4) (Exterior derivative) We define $d : \Omega^k(X; V) \to \Omega^{k+1}(X; V)$ by

$$d\alpha(v_0, \dots, v_k) = \sum_{i=0}^{k}(-1)^k v_i\alpha(v_0, \dots \hat{v}_i \dots v_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j}\alpha([v_i, v_j], v_0 \dots \hat{v}_i \dots \hat{v}_j \dots, v_k).$$

This squares to zero (i.e. $d^2\alpha = 0$) and satisfies

i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (d\beta)$
ii) $d\kappa(\alpha \wedge \beta) = \kappa(d\alpha \wedge \beta) + (-1)^{|\alpha|}\kappa(\alpha \wedge d\beta)$.

Let $A \in \Omega^1(P) \otimes \mathfrak{g}$ be a connection on P. By taking a trivialising cover $\{U_\alpha\}$, we obtain local $\mathfrak{g}$-valued 1-forms $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$. Then $\rho_*(A_\alpha) \in \Omega^1(U_\alpha) \otimes \mathrm{End}(V)$ are local $\mathrm{End}(V)$-valued forms on X. We claim that these local 1-forms define an 'affine connection' on the associated vector bundle $P(V)$; this claim is proved in Lemma 2.9 below, but first we give the definition of an affine connection.

Definition 2.7. Let E be a vector bundle over X with fibre V. An affine connection is a linear map $\nabla : \Omega^0(X, E) \to \Omega^1(X, E)$ satisfying the Liebniz rule $\nabla(fs) = \nabla(s) \otimes f + df \otimes s$ for $s \in \Omega^0(X, E)$ and $f \in \mathcal{C}^\infty(X)$.

Remark 2.8. An affine connection $\nabla : \Omega^0(X, E) \to \Omega^1(X, E)$ can be extended using the Liebniz rule to a connection $\nabla : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$.

Given a cover U_α of X over which E is trivial, we construct local 1-forms $\nabla_\alpha \in \Omega^1(U_\alpha) \otimes \mathrm{End}(V)$ as follows. Take a local frame $s_1, \dots s_n$ of E over U_α; then $\nabla(s_i) = \sum_{j=1}^n (\nabla_\alpha)_{ij} s_j$ for some local 1-forms $(\nabla_\alpha)_{ij} \in \Omega^1(U_\alpha)$ and define $\nabla_\alpha = \{(\nabla_\alpha)_{ij}\} \in \Omega^1(U_\alpha) \otimes \mathrm{End}(V)$. The transformation rule for the ∇_α on $U_\alpha \cap U_\beta$ is

$$\nabla_\beta = h_{\alpha\beta}^{-1} \nabla_\alpha h_{\alpha\beta} + h_{\alpha\beta}^{-1} dh_{\alpha\beta} \tag{4}$$

where $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ denote the transition functions on E. Similarly, a collection of local $\mathrm{End}(V)$-valued 1-forms $\nabla_\alpha \in \Omega^1(U_\alpha) \otimes \mathrm{End}(V)$ defined on an open cover $\{U_\alpha\}$ of X on which E is trivial that satisfy this transformation rule define an affine connection on E.

Lemma 2.9. *Let P be a principal G-bundle on X with and $\rho : G \to \mathrm{GL}(V)$ be a representation. Then a connection A on P induces an affine connection $\nabla_A : \Omega^0(X, P(V)) \to \Omega^1(X, P(V))$ on the associated vector bundle $P(V) = P \times_G V$.*

Proof. Take a cover U_α of X over which P is trivial and local sections $s_\alpha : U_\alpha \to P$. Then the local 1-forms $A_\alpha = s_\alpha^* A \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ transform as at (2). This open cover is also a cover on which $P(V)$ is trivial and the transition functions for $P(V)$ are $h_{\alpha\beta} = \rho \circ g_{\alpha\beta}$. We define local 1-forms $\nabla_\alpha := \rho_*(A_\alpha) \in \Omega^1(U_\alpha) \otimes \mathrm{End}(V)$ and observe that the transformation rule (4) can be verified by applying ρ_* to (2). Hence the local 1-forms ∇_α define an affine connection ∇_A on $P(V)$. □

This associated connection ∇_A is known as the covariant derivative d_A of A and can instead be constructed using an equivariant horizontal distribution on P.

Definition 2.10. Let $\pi : P \to X$ be a principal G-bundle with connection A and $\rho : G \to \mathrm{GL}(V)$ a representation. We make the following definitions.

(1) A V-valued k-form α on P is G-invariant if $R_g^* \alpha = \rho(g^{-1}) \circ \alpha$ for all $g \in G$.

Let $\Omega^k(P; V)^G$ denote the space of G-invariant V-valued k-forms on P.

(2) A V-valued k-form α on P is horizontal if $\alpha(\eta_1, \dots, \eta_k) = 0$ whenever any η_i is a vertical tangent vector.

Let $\Omega_h^k(P; V)$ denote the space of horizontal V-valued k-forms on P.

The exterior derivative d preserves G-invariant forms but not horizontal forms. Instead, if $h_A : TP \to TP$ denotes the projection onto the horizontal tangent space defined by A, we can use the exterior derivative and the projection h_A to construct a covariant derivative d_A on $\Omega_h^\bullet(P; V)^G$. In general this does not square to zero and so is not a differential.

Definition 2.11. Let A be a principal connection on a principal bundle G-bundle P and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on a vector space V. Then the associated covariant derivative is the map $d_A : \Omega_h^k(P; V)^G \to \Omega_h^{k+1}(P; V)^G$ defined by

$$d_A(\beta)(\xi_1, \dots, \xi_{k+1}) = d\beta(h_A(\xi_1), \dots, h_A(\xi_{k+1}))$$

for vector fields ξ_i.

Lemma 2.12. *Let $\pi : P \to X$ be a principal G-bundle with connection A and $\rho : G \to \mathrm{GL}(V)$ a representation; then*
$$\Omega^k_h(P;V)^G \cong \Omega^k(X, P(V)).$$

Proof. We take a cover U_α of X on which P is trivial and corresponding local sections $s_\alpha : U_\alpha \to P$ with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Given a V-valued k-form ω on P, we define $\omega_\alpha := s_\alpha^* \omega \in \Omega^1(U_\alpha) \otimes V$. Such local k-forms ω_α define a $P(V)$-valued k-form on X if and only if they satisfy the transformation law
$$\omega_\beta = \rho(g_{\alpha\beta}^{-1})\omega_\alpha.$$
This is true if and only if the V-valued k-form ω on P is horizontal and G-invariant. □

Then, via this isomorphism, we can instead view the covariant derivative as a map
$$d_A : \Omega^k(X, V(P)) \to \Omega^{k+1}(X, P(V)).$$

Example 2.13. Consider the adjoint bundle $\mathrm{ad}P := P \times_G \mathfrak{g}$ associated to the adjoint representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ and a principal G-bundle $P \to X$. The Lie bracket $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and the wedge product on forms can be used to give $\Omega^\bullet(X, \mathrm{ad}P)$ a graded Lie algebra structure on $\Omega^\bullet(X, \mathrm{ad}P)$. More precisely, we can define
$$[- \wedge -] : \Omega^k(X, \mathrm{ad}P) \times \Omega^l(X, \mathrm{ad}P) \to \Omega^{k+l}(X, \mathrm{ad}P)$$
as in the third part of Definition 2.6. This satisfies:
(1) $[\alpha \wedge \beta] = (-1)^{|\alpha||\beta|+1}[\beta \wedge \alpha]$;
(2) $(-1)^{|\alpha||\gamma|}[\alpha \wedge [\beta \wedge \gamma]] + (-1)^{|\alpha||\beta|}[\beta \wedge [\gamma \wedge \alpha]] + (-1)^{|\beta||\gamma|}[\gamma \wedge [\alpha \wedge \beta]] = 0$.

Given a G-connection A on P, we can consider the covariant derivative
$$d_A : \Omega^k(X, \mathrm{ad}P) \to \Omega^{k+1}(X, \mathrm{ad}P)$$
that satisfies $d_A\beta = d\beta + [A, \beta]$. If, moreover, $d_A^2 = 0$, then the exterior derivative can be viewed as a differential giving $\Omega^\bullet(X, \mathrm{ad}P)$ the structure of a differential graded Lie algebra. The obstruction to d_A being a differential on $\Omega^\bullet(X, \mathrm{ad}P)$ is measured by the curvature form of A which is defined in §2.3.

2.2. Connections on associated principal bundles.

Definition 2.14. Let P be a principal G-bundle over X and A a connection on P with associated local 1-forms $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$.
(1) For a group homomorphism $\rho : G \to H$, there is an induced connection $\rho_* A$ on the associated H-bundle $H(P) := P \times_G H$ (defined by local 1-forms $\rho_* A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{h}$).
(2) For a closed subgroup $H \subset G$, a reduction of P to H is a principal H-bundle $P_H \subset P$.
(3) A connection A on P is extended from a connection A_H on a reduction P_H of P to a closed subgroup $\iota : H \hookrightarrow G$ if the induced connection $\iota_*(A_H)$ on $P_H \times_H G$ coincides with A under the natural isomorphism $P_H \times_H G \cong P$.

Lemma 2.15. *Let P be a principal G-bundle over X and $H \subset G$ a closed subgroup; then a reduction of P to H is equivalent to a section $\sigma : X \to P/H$.*

Proof. We use the fact that $P \to P/H$ is a principal H-bundle. If $P_H \subset P$ is a reduction to H, the composition $P_H \hookrightarrow P \to P/H$ is H-invariant i.e. constant along the fibres of $P_H \to X$. Hence it descends to a section $\sigma : X \to P/H$. Conversely, given a section $\sigma : X \to P/H$, we can pullback the principal H-bundle $P \to P/H$ along σ to obtain the reduction $P_H := \sigma^* P$. □

2.3. **Curvature.** Let $A \in \Omega^1(P) \otimes \mathfrak{g}$ be a connection on a principal G-bundle $\pi : P \to X$. Consider the covariant derivative
$$d_A := d \circ h_A : \Omega^k(X, \mathrm{ad}P) \to \Omega^{k+1}(X, \mathrm{ad}P)$$
associated to the adjoint representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ and the connection A.

Definition 2.16. The curvature 2-form of a connection A is $F_A := d_A(A) \in \Omega^2(X, \mathrm{ad}P)$.

Often one defines the curvature form $F_A \in \Omega^2(P) \otimes \mathfrak{g}$ by its structural equation

$$F_A = dA + \frac{1}{2}[A, A]$$

where, for vector fields ξ and ψ on P, we define $[A, A](\xi, \psi) = 2[A(\xi), A(\psi)]$ using the Lie bracket on $\mathfrak{g}$. Then one must check that F_A is horizontal and G-invariant so that $F_A \in \Omega^2(X, \mathrm{ad}P)$.

Lemma 2.17. *For a connection A on a principal G-bundle $\pi : P \to X$, the form*

$$F_A := dA + \frac{1}{2}[A, A] \in \Omega^2(P) \otimes \mathfrak{g}$$

defines an adP-valued 2-form on X which we also denote by F_A.

Proof. We produce local $\mathfrak{g}$-valued 2-forms on X and verify these define a globally defined $\mathrm{ad}P$-valued 2-form on X (in the spirit of Lemma 2.12). As usual, we take local trivialising sections $s_\alpha : U_\alpha \to P$ with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Let $A_\alpha := s_\alpha^* A \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ and $F_\alpha = s_\alpha^* F_A \in \Omega^2(U_\alpha) \otimes \mathfrak{g}$; then also

$$F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha].$$

It follows from the transformation law (2) for the local 1-forms A_α that we have

$$\begin{aligned} F_\beta &= dA_\beta + \frac{1}{2}[A_\beta, A_\beta] \\ &= d(\mathrm{ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^*\theta) + \frac{1}{2}[\mathrm{ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^*\theta, \mathrm{ad}(g_{\alpha\beta}^{-1})A_\alpha + g_{\alpha\beta}^*\theta] \\ &= \mathrm{ad}(g_{\alpha\beta}^{-1})dA_\alpha + g_{\alpha\beta}^* d\theta + \frac{1}{2}\mathrm{ad}(g_{\alpha\beta}^{-1})[A_\alpha, A_\alpha] + \frac{1}{2}g_{\alpha\beta}^*[\theta, \theta] \\ &= \mathrm{ad}(g_{\alpha\beta}^{-1})F_\alpha \end{aligned} \tag{5}$$

where the final line follows from the Maurer-Cartan equation for θ. Hence, the local 2-forms $F_\alpha = s_\alpha^* F_A \in \Omega^2(U_\alpha) \otimes \mathfrak{g}$ glue to give a global 2-form $F_A \in \Omega^2(X, \mathrm{ad}P)$ (cf. Lemma 2.12). □

One final alternative definition of the curvature 2-form is as the composition

$$F_A := d_A^2 : \Omega^0(X, \mathrm{ad}P) \to \Omega^2(X, \mathrm{ad}P);$$

in this case, the Leibniz rule for d_A is used to verify that F_A is $\mathcal{C}^\infty(X)$-linear and so defines a global 2-form $F_A \in \Omega^2(X, \mathrm{ad}P)$. We check that this definition coincides with the structural equation definition:

$$\begin{aligned} d_A^2(\beta) &= d_A(d\beta + [A, \beta]) = d^2\beta + d[A, \beta] + [A, d\beta] + [A, [A, \beta]] \\ &= [dA + \frac{1}{2}[A, A], \beta] = [F_A, \beta]. \end{aligned}$$

Lemma 2.18. *Let F_A be the curvature form of a connection A on a principal bundle P. Then:*

(1) *For $g \in G$, we have $R_g^* F_A = ad(g^{-1})F_A$;*
(2) *$dF_A = [F_A, A]$ (Bianci identity).*

Proof. For the first statement, we note that $[\mathrm{ad}(g^{-1})A, \mathrm{ad}(g^{-1})A] = \mathrm{ad}(g^{-1})[A, A]$ and also $d(\mathrm{ad}(g^{-1})A) = \mathrm{ad}(g^{-1})dA$. For the second statement, we have

$$dF_A = d^2 A + \frac{1}{2}d[A, A] = \frac{1}{2}\left([dA, A] - [A, dA]\right) = [dA, A] = -\frac{1}{2}[[A, A], A] + [F_A, A] = [F_A, A].$$

□

Definition 2.19. A connection A on P is flat if its curvature form is zero: $F_A = 0$.

Remark 2.20. The curvature form only gives local information. If P is flat we cannot in general say whether P is trivial or even has trivial holonomy. For example, consider the (non-smooth) example $\pi : \mathbb{R} \to S^1$ given by $\pi(r) = \exp(2\pi i r)$ which is a principal $\mathbb{Z}$-bundle. As the fibres are zero dimensional, the only choice of connection is $A = 0$ which is a flat connection. However, the bundle is not trivial and does not have trivial holonomy.

3. Gauge theory for principal bundles

In this section, we review the gauge theoretic set up for the principal G-bundle $\pi : P \to X$ where G is a compact Lie group and X is a Riemann surface. The standard reference for this material is the fundamental paper of Atiyah and Bott [1].

Let $\mathcal{A} = \mathcal{A}_P$ denote the set of G-connections on P. As the difference of any two G-connections $A, A' \in \mathcal{A}$ is horizontal:

$$(A - A')_p(\rho_p(\eta)) = A_p(\rho_p(\eta)) - A'_p(\rho_p(\eta)) = \eta - \eta = 0,$$

we see that $\mathcal{A}$ is an infinite dimensional affine space modeled on

$$(\Omega^1_h(P) \otimes \mathfrak{g})^G \cong \Omega^1(X, \mathrm{ad}P).$$

The space of G-connections $\mathcal{A}$ admits a symplectic structure ω described as follows: for $A \in \mathcal{A}$, the symplectic form ω_A on $T_A\mathcal{A} \cong (\Omega^1_h(P) \otimes \mathfrak{g})^G$ is defined by combining a fixed G-invariant inner product $\kappa(-,-)$ on $\mathfrak{g}$ with the wedge product on forms on P. More precisely, the inner product κ and wedge product $\wedge$ define a skew-symmetric bilinear map

$$\kappa(- \wedge -) : (\Omega^1_h(P) \otimes \mathfrak{g})^G \times (\Omega^1_h(P) \otimes \mathfrak{g})^G \to \Omega^2_h(P)^G \cong \Omega^2(X)$$

where $\kappa(\alpha \wedge \beta)(\xi_1, \xi_2) = \kappa(\alpha(\xi_1), \beta(\xi_2)) - \kappa(\alpha(\xi_2), \beta(\xi_1))$. We define the symplectic form ω_A on $(\Omega^1_h(P) \otimes \mathfrak{g})^G$ to be the composition of this map with the integration map $\Omega^2(X) \to \mathbb{R}$; thus,

$$\omega_A(\beta, \gamma) := \int_X \kappa(\beta \wedge \gamma).$$

Remark 3.1. A symplectic structure on an infinite dimensional manifold $\mathcal{A}$ is a closed 2-form ω on $\mathcal{A}$ that is non-degenerate in the sense that $\omega_A : T_A\mathcal{A} \to T^*_A\mathcal{A}$ is injective for all $A \in \mathcal{A}$.

3.1. **The gauge group action.** The gauge group $\mathcal{G} = \mathcal{G}_P$ is the group of automorphisms of P (as a principal G-bundle) and we refer to elements $\Phi : P \to P$ of the gauge group as gauge transformations. Equivalently, a gauge transformation can be thought of as a G-equivariant homomorphism $f : P \to G$ or a section of the associated fibre bundle $\mathrm{Ad}P := P \times_G G$ where G acts on itself by conjugation. Hence we make use of the following identifications:

$$\mathcal{G} = \mathrm{Aut}^G(P) = \mathrm{Hom}^G(P, G) = \Gamma(\mathrm{Ad}P).$$

The group structure is given by composition of automorphisms and we often simply write $\Phi\Psi$ to mean the composition $\Phi \circ \Psi$.

The gauge group acts on the space of G-connections on P by pulling back a G-connection $A \in \Omega^1(P) \otimes \mathfrak{g}$ along a gauge equivalence $\Phi : P \to P$; we write this as a left action so

$$\Phi \cdot A = (\Phi^{-1})^* A.$$

We refer to the orbits of the gauge group action as gauge equivalence classes.

Remark 3.2. If we instead think of the connection 1-form A as a G-equivariant horizontal distribution $H = \{H_p \subset T_pP\}$, then the action is given by $\Phi \cdot H = \{((\Phi^{-1})^*H)_{\Phi(p)} \subset T_pP\}$.

As $\mathcal{G} = \Gamma(\mathrm{Ad}P)$, its Lie algebra $\mathrm{Lie}\,\mathcal{G}$ can naturally be identified with the space of sections $\Gamma(\mathrm{ad}P)$. The infinitesimal gauge action is defined as follows: for $\Psi \in \mathrm{Lie}\,\mathcal{G}$ and $A \in \mathcal{A}$, we have

$$\Psi_A := \frac{d}{dt}(-\mathrm{exp} t\Psi) \cdot A|_{t=0} = -d_A\Psi \in T_A\mathcal{A}.$$

The Lie bracket on $\mathrm{Lie}\,\mathcal{G}$ which comes from the Lie bracket on $\mathfrak{g}$. In fact $\Omega^\bullet(X, \mathrm{ad}P)$ has a graded Lie algebra structure with bracket

$$[-,-]_\mathfrak{g} : \Omega^p(X, \mathrm{ad}P) \times \Omega^q(X, \mathrm{ad}P) \to \Omega^{p+q}(X, \mathrm{ad}P)$$

formed by combining the Lie bracket on $\mathfrak{g}$ and the wedge product on forms (see Example 2.13). The topologically duality pairing $\Omega^0(X) \times \Omega^2(X) \to \mathbb{R}$ given by

$$(f, \beta) \mapsto \int_X f\beta$$

and the inner product κ on $\mathfrak{g}$, give rise to an isomorphism $\Omega^2(X, \mathrm{ad}P) \cong \Omega^0(X, \mathrm{ad}P)^* = \mathrm{Lie}\,\mathcal{G}^*$.

The moment map for the gauge group action is given by taking the curvature of a connection (modulo a sign):

$$\begin{aligned} \mu : \mathcal{A} &\to \Omega^0(X, \mathrm{ad}P)^* = \mathrm{Lie}\, \mathcal{G}^* \\ A &\mapsto -F_A \end{aligned}$$

where we abuse notation and write F_A to mean both the connection 2-form of A in $\Omega^2(X, \mathrm{ad}P)$ and also its image in $\mathrm{Lie}\, \mathcal{G}^*$ under the isomorphism $\Omega^2(X, \mathrm{ad}P) \cong \Omega^0(X, \mathrm{ad}P)^*$. The moment map is equivariant and the presence of the minus sign in the above definition of μ is required in order for us to verify the moment map condition:

$$d_A\mu(\beta) \cdot \Psi = \omega_A(\Psi_A, \beta) \tag{6}$$

for $A \in \mathcal{A}$ and $\beta \in T_A\mathcal{A} \cong \Omega^1(X, \mathrm{ad}P)$ and $\Psi \in \mathrm{Lie}\, \mathcal{G} = \Omega^0(X, \mathrm{ad}P)$. More precisely,

$$d_A\mu(\beta) = \frac{d}{dt}\mu(A + t\beta)|_{t=0} = \frac{d}{dt}(-F_A - td_A)\beta|_{t=0} = -d_A(\beta)$$

and then, using integration by parts, we have

$$d_A\mu(\beta) \cdot \Psi := -\int_X \kappa(d_A\beta \wedge \Psi) = \int_X \kappa(\beta \wedge d_A\Psi) = \int_X \kappa(\Psi_A \wedge \beta) = \omega_A(\Psi_A, \beta).$$

3.2. **Characteristic classes of principal bundles.** Topologically, a principal G-bundle P over a Riemann surface X is characterised by its characteristic classes. In this section, we explain the Chern-Weil description of characteristic classes which uses a connection on P. Given a G-invariant linear map $\chi : \mathfrak{g} \to \mathbb{R}$, we can associate to a principal G-bundle P a characteristic class

$$\chi(P) \in H^2(X, \mathbb{R})$$

as follows. By taking the composition of χ with the curvature form F_A of any principal connection A on P, we obtain a 2-form $\chi(A) \in \Omega^2(P)$ defined by $\chi(A)_p = \chi \circ (F_A)_p$. As the curvature F_A is horizontal, it follows that $\chi(A)$ is also horizontal. Similarly, since χ is G-invariant and F_A is invariant, $\chi(A)$ is G-invariant and so $\chi(A)$ is the pullback under $\pi : P \to X$ of a 2-form on X which we also denote by $\chi(A)$:

$$\chi(A) \in \Omega^2(X) \cong \Omega^2_h(P)^G.$$

As X is two dimensional, the two form $\chi(\alpha)$ is closed and so defines a cohomology class

$$[\chi(A)] \in H^2_{dR}(X, \mathbb{R}).$$

Moreover, if A' is another G-connection on P, then $\chi(A) - \chi(A')$ is an exact 2-form (this is proved by producing a 'homotopy' between these connections). Hence, the cohomology class

$$\chi(P) := [\chi(A)] \in H^2_{dR}(X, \mathbb{R})$$

is independent of the choice of connection on P. We call the class $\chi(P)$ the characteristic class of P associated to χ. Since linear maps $\chi : \mathfrak{g} \to \mathbb{R}$ are in bijective correspondence with characters of G, we get a map

$$c(P) : \mathcal{X}^*(G) \to H^2(X, \mathbb{R});$$

or, equivalently, we can interpret this as a class

$$c(P) \in H^2(X, \mathcal{X}_*(G)) = H^2(X, \pi_1(G)) \cong \pi_1(G)$$

which we call the characteristic class of P.

Fix a metric on X whose associated volume form induces the given orientation on X; then there is an associated Hodge star operator $\star : \Omega^k(X) \to \Omega^{2-k}(X)$.

Definition 3.3. A G-connection A on P is

(1) flat if $F_A = 0$;
(2) projectively flat if $\star F_A \in \Omega^0(X, \mathrm{ad}P) = \mathrm{Hom}^G(P, \mathfrak{g})$ is a constant element in the centre of $\mathfrak{g}$.

Remark 3.4. In particular, if a principal G-bundle P over a Riemann surface admits a flat connection, then all its characteristic classes vanish.

Example 3.5. Let $G = \mathrm{U}(n)$; then in this case there is only one character, the determinant, whose derivative $\chi : \mathfrak{u}(n) \to \mathbb{R}$ is then the (scaled) trace map

$$A \mapsto \frac{i}{2\pi}\mathrm{Tr}(A).$$

If P is a principal $\mathrm{U}(n)$-bundle and we let $E = P \times_{\mathrm{U}(n)} \mathbb{C}^n$ denote the associated complex vector bundle, then the characteristic class of P corresponds to the first chern class of E:

$$\chi(P) = c_1(E) \in H^1(X, \mathbb{R}).$$

If there is a connection A on P such that $\star F_A = -i\mu I \in \mathfrak{u}(n)$ for some real number μ (i.e. A is projectively flat), then

$$\deg(E) := \int_X c_1(E) = \int_X \frac{i}{2\pi}\mathrm{Tr}(F_A) = \int_X \frac{i}{2\pi}\mathrm{Tr}(\star F_A)\omega = \int_X \frac{1}{2\pi}\mathrm{Tr}(\mu I)\omega = \frac{n\mu}{2\pi}\int_X \omega.$$

In particular, if we normalise our metric on X so that $\int_X \omega = 2\pi$, then we see that

$$\mu = \mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}$$

is the slope of the vector bundle E. Therefore, the real number μ associated to a projectively flat connection on P is (up to normalisation) the slope of the complex vector bundle associated to P.

3.3. **The gauge theoertic moduli spaces.** The moduli space of gauge equivalence classes of flat G-connections on P is constructed as the symplectic reduction

$$\mathcal{M} = \mathcal{M}_P(X, G) := \mu^{-1}(0)/\mathcal{G};$$

that is, the space of gauge equivalence classes of flat connections. If a principal G-bundle P admits a flat connection, then all its characteristic classes vanish (cf. §3.2). More generally, we can consider the moduli space of gauge equivalence classes of projectively flat connections

$$\mathcal{M} = \mathcal{M}_P(X, G) := \mu^{-1}(C)/\mathcal{G}$$

for a central value $C \in \mathfrak{g}$ (where here we also view C as a constant map in $\mathrm{Hom}^G(P, \mathfrak{g}) = \mathrm{Lie}\,\mathcal{G} \cong \mathrm{Lie}\,\mathcal{G}^*$). In fact we will use $\mathcal{M} = \mu^{-1}(C)/\mathcal{G}$ to denote both the moduli space of flat and projectively flat connections and allow the choice of $C = 0$ when we want to consider flat connections. In nice cases these symplectic reductions are smooth complex manifolds which inherit a symplectic structure from that of $\mathcal{A}$; more generally, they can be described as stratified symplectic spaces. These moduli spaces of connections are often referred to as the de Rham moduli spaces and denoted $\mathcal{M}_{\mathrm{dR}}(X, G)$.

3.4. **The Yang-Mills functional.** The fixed metric on X and inner product κ on $\mathfrak{g}$ can be used to define an inner product on $\mathrm{Lie}\,\mathcal{G}$ which is invariant under the adjoint action of $\mathcal{G}$ on $\mathrm{Lie}\,\mathcal{G}$ as follows. We can construct an inner product on $\Omega^k(X)$ by

$$(\alpha, \beta) = \int_X \alpha \wedge \star\beta$$

and similarly, by also using the inner product κ on $\mathfrak{g}$, an inner product on $\Omega^k(X, \mathrm{ad}P)$:

$$(\alpha, \beta)_\kappa = \int_X \kappa(\alpha \wedge \star\beta).$$

Finally, the Yang-Mills functional is defined to be the norm square of the moment map; that is,

$$S_{YM} : \mathcal{A} \to \mathbb{R}$$
$$S_{YM}(A) := ||F_A||^2 = \frac{1}{2}\int_X \kappa(F_A \wedge \star F_A).$$

Definition 3.6. A connection $A \in \mathcal{A}$ is a Yang-Mills connection if it is a critical point of the Yang-Mills functional.

Lemma 3.7. *A connection $A \in \mathcal{A}$ is Yang-Mills if and only if $d_A \star F_A = 0$.*

Proof. For $\tau \in \Omega^1(X, \mathrm{ad}P)$, we have:

$$F_{A+t\tau} = F_A + td_A\tau + \frac{1}{2}t^2[\tau,\tau]$$

$$S_{YM}(A+t\tau) = S_{YM}(A) + t\int_X \kappa(d_A\tau \wedge \star F_A) + \text{ higher order terms}$$

(cf. [1] §4). Thus A is critical for the Yang-Mills functional if and only if

$$\frac{d}{dt}S_{YM}(A+t\tau)|_{t=0} = 0 \iff \kappa(d_A\tau \wedge \star F_A) = 0 \iff \kappa(\tau \wedge d_A \star F_A) = 0,$$

for all $\tau \in \Omega^1(X, \mathrm{ad}P)$. Hence A is Yang-Mills if and only if $d_A \star F_A = 0$. □

The Yang-Mills functional can be used to produced a Morse type flow for points in $\mathcal{A}$ as follows. The exterior derivative of the Yang-Mills functional, gives a 1-form on $\mathcal{A}$ defined by

$$\begin{array}{rccc} d_A S_{YM}: & T_A\mathcal{A} & \to & \mathbb{R} \\ & \tau & \mapsto & \int_X \kappa(d_A\tau \wedge \star F_A) \end{array}.$$

The gradient vector field associated to the Yang-Mills functional is then defined by

$$\mathrm{grad}_A S_{YM} = \star d_A \star F_A = -d_A^* F_A$$

where $d_A^* = -\star d_A \star$ is the adjoint to $d_A : \Omega^k(X, \mathrm{ad}P) \to \Omega^{k+1}(X, \mathrm{ad}P)$. This gives the Yang-Mills flow. By definition, the stationary points of the Yang-Mills flow are the Yang-Mills connections. We also remark that the Yang-Mills flow exists for finite time and the Yang-Mills flow stays in a complex gauge orbit; for further properties of the Yang-Mills flow, see [2].

4. Representations of the fundamental group

Let $\tilde{X}$ be the universal cover of X; then the fundamental group $\pi_1(X)$ acts freely by deck transformations and the quotient $p : \tilde{X} \to X$ is a principal $\pi_1(X)$-bundle. Given a representation $\rho : \pi_1(X) \to G$, we can construct an associated G-bundle

$$P_\rho = \tilde{X} \times_{\pi_1(X)} G$$

over X. As $\pi_1(X)$ is given the discrete topology, the transition functions for $p : \tilde{X} \to X$ are locally constant. Hence, all associated principal G-bundle, such as P_ρ, have locally constant transition functions too.

Definition 4.1. A principal G-bundle $\pi : P \to X$ is a G-local system if there is a cover $\{U_\alpha\}$ of X on which P is trivial with locally constant transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

As $p : \tilde{X} \to X$ has locally constant transition functions, we can define a flat connection on this principal $\pi_1(X)$-bundle. This gives rise to a flat connection on P_ρ. In particular, a principal G-bundle arising from a representation $\rho : \pi_1(X) \to G$ is a G-local system and admits a flat connection. In fact, we will see that the converse of this statement is true and so we get an equivalence, known as the Riemann–Hilbert correspondence, between G-local systems, flat connections on principal G-bundles and representations of the fundamental group in G.

4.1. **Holonomy of a connection.**

Let $\pi : P \to X$ be a principal G-bundle and let A be a connection on P. We assume our base X is paracompact and connected.

Definition 4.2. A piecewise continuous path $\gamma : [0,1] \to P$ is said to be horizontal with respect to A if, for $t \in [0,1]$, the tangent vectors $\gamma'(t) \in T_{\gamma(t)}P$ are horizontal.

Lemma 4.3. *Let A be a connection on a principal G-bundle $\pi : P \to X$ and $\gamma : [0,1] \to X$ be a piecewise continuous path. For $p \in P$ such that $\pi(p) = \gamma(0)$, there is a unique horizontal lift $\tilde{\gamma}_p : [0,1] \to P$ starting at p; that is, $\tilde{\gamma}_p$ is horizontal and $\pi \circ \tilde{\gamma}_p = \gamma$.*

Proof. We can cover the image of γ by open sets U_i on which P is trivial and so without loss of generality we can assume γ has image in an open set $U \subset X$ over which P is trivial. To lift the loop γ in U to $P|_U \cong U \times G$, we want to find $g : [0,1] \to G$ and then define $\tilde{\gamma}_p(t) := \varphi(\gamma(t), g(t))$ where $\varphi : U \times G \to P|_U$ is the above trivialising isomorphism. The lift starts is horizontal if and only if $g'(t) + A(\gamma'(t))g(t) = 0$. This gives a first order differential equation for g with initial condition $p = \varphi(\gamma(0), g(0))$ and so there is a unique solution. □

Definition 4.4. The parallel transport of a piecewise continuous path $\gamma : [0,1] \to X$ with respect to a connection A on a principal G-bundle P is the map

$$\begin{array}{cccc} P_\gamma^A : & P_{\gamma(0)} & \to & P_{\gamma(1)} \\ & p & \mapsto & \tilde{\gamma}_p(1) \end{array}$$

where $\tilde{\gamma}_p : [0,1] \to P$ is the unique horizontal lift of γ starting at p. As the connection A is G-invariant, the parallel transports satisfy

$$P_\gamma^A \circ R_g = R_g \circ P_\gamma^A.$$

If $\gamma : [0,1] \to X$ is a loop at $x \in X$ and $p \in \pi^{-1}(x)$, we define the holonomy at p of γ with respect to A to be the element $\mathrm{Hol}_p(A, \gamma)$ of G such that

$$p \cdot \mathrm{Hol}_p(A,\gamma) = P_\gamma^A(p).$$

This defines the holonomy map

$$\mathrm{Hol}_p(A, -) : \{\text{Loops } \gamma : [0,1] \to X \text{ at } x\} \to G$$

whose image is the holonomy group $H_p(A)$ of A at p.

We give without proof some basic properties of the holonomy of a connection A on P (for example, see [7] Chapter 1 $S4$)

Lemma 4.5. *Let $\gamma, \gamma' : [0,1] \to X$ be loops at $x \in X$ and $\delta : [0,1] \to X$ a path from x to $y \in X$. Then, for $p \in \pi^{-1}(x)$, the following statements hold.*

(1) $\mathrm{Hol}_{pg}(A,\gamma) = g^{-1} \mathrm{Hol}_p(A,\gamma) g$.
(2) $\mathrm{Hol}_p(A,\gamma) = \mathrm{Hol}_{P_{\delta(A)(p)}}(A, \delta \cdot \gamma \cdot \delta^{-1})$.
(3) $\mathrm{Hol}_p(A, \gamma \cdot \gamma') = \mathrm{Hol}_p(A,\gamma') \mathrm{Hol}_p(A,\gamma)$.

In particular, this lemma implies that the map

$$\rho_{A,p} : \{\text{Loops } \gamma : [0,1] \to X \text{ at } x\} \to G$$

defined by $\gamma \mapsto \mathrm{Hol}_p(A,\gamma)^{-1}$ is a group homomorphism. Our aim is to show that, for a flat connection A, the homomorphism $\rho_{A,p}$ descends to give a representation of the fundamental group of X. For this, we need the following lemma.

Lemma 4.6. *Let A be a flat connection on a principal G-bundle $\pi : P \to X$; then any contractible loop γ at $x \in X$ has trivial holonomy; that is,* $\mathrm{Hol}_p(A,\gamma) = e$.

Proof. We fix $p \in \pi^{-1}(x)$ and consider the set of points reached by parallel transporting p along paths starting at x:

$$P_A(p) := \{P_\delta^A(p) \in P : \delta \text{ is a path starting at } x\}.$$

We claim that $P_A(p)$ is a submanifold of P. To see this we recall that, as A is flat, the corresponding horizontal subspaces give rise to a foliation of P. Hence, in a neighbourhood U of x there is a unique horizontal section s of $P|_U$ such that $s(x) = p$. Therefore, for $q \in P_A(p)$, there is a chart in a neighbourhood U_q of q in $P_A(p)$ of dimension equal $\dim X + \dim H_p(A)$. Hence $P_A(p)$ is a submanifold of P (in fact, this is a consequence of the Reduction Theorem given in §4.2 below).

For a flat connection A, the holonomy group $H_p(A)$ is discrete (cf. [7] Chapter 1, §9) and so $\pi|_{P_A(p)} : P_A(p) \to X$ is a covering space. The contracible loop γ in X, then lifts to a loop in the covering space $P_A(p)$ and so

$$p = P_\gamma^A(p) = p \cdot \mathrm{Hol}_p(A,\gamma);$$

that is, the holonomy around γ is trivial. □

For a flat connection A on P and a point $p \in \pi^{-1}(x)$, we can construct, by the above lemma, an induced homomorphism

$$\rho_{A,p} : \pi_1(X, x) \to G \qquad [\gamma] \mapsto \mathrm{Hol}_p(A, [\gamma])^{-1}.$$

The image of this homomorphism is by definition the homology group $H_A(p)$. If we choose a different point $p \cdot g$ in the fibre over x, then the associated representation is just a conjugation of this representation by Lemma 4.5. Hence this representation is defined by the connection A up to conjugation. We can now relate flat connections to representations of the fundamental group.

Theorem 4.7. *There is a bijection between the set of conjugacy classes of representations $\rho : \pi_1(X, x) \to G$ and the set of gauge equivalence classes of flat connections on principal G-bundles over X.*

Proof. Given a flat connection A on a principal G-bundle P, we can fix $p \in \pi^{-1}(x)$ and consider the holonomy representation

$$\rho_{A,p} : \pi_1(X, x) \to G.$$

This defines a map from the set of flat connections on principal G-bundles over X to the set of representations of $\pi_1(X, x)$ in G. The induced map from the set of flat connections on principal G-bundles over X to the set of conjugacy classes of such representations is independent of the choice of $p \in \pi^{-1}(x)$ by the discussion preceding this theorem. Then to prove the theorem, we need to show:

(1) this is surjective;
(2) two flat connections are in the kernel if and only if they are gauge equivalent.

For surjectivity, given a representation $\rho : \pi_1(X, x) \to G$, we can take the trivial G-bundle over the universal cover $\tilde{X}$ of X with trivial flat connection $(\tilde{X} \times G, \tilde{A})$. The fundamental group acts on the universal cover $\tilde{X}$ by deck transformations and the quotient is simply $\tilde{X} \to X$. Using the representation ρ, we construct an action of $\pi_1(X, x)$ on $\tilde{X} \times G$ as follows

$$[\gamma] \cdot (\tilde{x}, g) = ([\gamma] \cdot \tilde{x}, \rho([\gamma]) \cdot g).$$

Then we construct a principal G-bundle P_ρ over X with flat connection A_ρ as the quotient of $(\tilde{X} \times G, \tilde{A})$ by the action of $\pi_1(X, x)$ described above.

If we have two flat connections on principal G-bundles over X, say (P, A) and (P', A'), that give the same holonomy representation conjugacy class i.e. $\rho_{A,p} = g\rho_{A',p'}g^{-1}$, then we claim that (P, A) and (P', A') are gauge equivalent. By replacing p' by $p' \cdot g$, we can assume that $\rho_{A,p} = \rho_{A',p}$. The idea is to construct a gauge equivalence $\Phi : P \to P'$ such that for all curves γ from x to y in X, we have

$$\Phi(P^A_\gamma(p)) = P^{A'}_\gamma(p').$$

This determines Φ on the subbundle of points that can be connected to p by a horizontal path (this is independent of the choice of path by the above lemma). To extend to the fibre over y we insist that Φ should be a G-bundle map and so $\Phi(P^A_\gamma(p) \cdot g) = P^{A'}_\gamma(p') \cdot g$. By construction Φ maps S-horizontal vectors in P to A'-horizontal vectors in P'; that is, Φ defines a gauge equivalence between (P, A) and (P', A'). □

4.2. **Reduction theorem.** For a connection A on a principal bundle P and $p \in P$, we let $P_A(p) \subset P$ be the set of points that can be connected to p by a horizontal curve. In this section, we state without proof the Reduction Theorem.

Theorem 4.8. *(Reduction theorem,* [7] *Chapter 1, Theorem 7.1) Let A be a connection on a principal G-bundle P over X and let $p \in P$ lie over $x \in X$; then $P_A(p) \subset P$ is a reduction of P to the holonomy subgroup $H_p(A) \subset G$. Furthermore, the connection A on P is reducible to a connection on $P_A(p)$.*

We also note the following (cf. [7], Chapter 1, Corollary 9.2).

Corollary 4.9. *Let A be a flat connection on a principal G-bundle P over a simply connected base X; then $P \cong X \times G$ is trivial and A is isomorphic to the trivial connection.*

4.3. **Riemann Hilbert for G-bundles.**

Proposition 4.10. *For a principal G-bundle $\pi : P \to X$, the following are equivalent.*

(1) *P admits a flat connection.*
(2) *P is a G-local system.*
(3) *$P \cong P_\rho$ for a representation $\rho : \pi_1(X) \to G$.*

Proof. At the beginning of this section, we saw the equivalence between flat connections and G-local systems. Finally, Theorem 4.7 gives the equivalence between flat connections and representations of the fundamental group. □

4.4. **The Betti moduli space.** Let X be a compact connected Riemann surface of genus g and G be a compact connected Lie group. The Betti moduli space (or G-character variety of X) is defined, as a set, to be the conjugacy classes of G-representations of the fundamental group $\pi_1(X)$; that is,

$$M_B(X,G) := \mathrm{Hom}(\pi_1(X),G)/G.$$

This set inherits a topology from G as follows. Take a presentation for $\pi_1(X)$

$$\pi_1(X) =< a_1, b_1, \cdots , a_g, b_g | \prod_{i=1}^{g} [a_i, b_i] = 1 >$$

and consider the map $\mu_g : G^{2g} \to G$ defined by $\mu_g(A_1, \dots , B_g) = \prod_{i=1}^g [A_i, B_i]$. Then the image of the natural inclusion $\mathrm{Hom}(\pi_1(X), G) \hookrightarrow G^{2g}$ given by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \dots , \rho(a_g), \rho(b_g))$$

is $\mu_g^{-1}(1)$; that is, $\mathrm{Hom}(\pi_1(X),G) = \mu_g^{-1}(1)$. We give the closed subset $\mathrm{Hom}(\pi_1(X),G) \subset G^{2g}$ the subspace topology and then $M_B(X,G)$ the quotient topology; i.e. a subset $U \subset M(X,G)_B$ is open if and only if $\pi^{-1}(U)$ is open in $\mathrm{Hom}(\pi_1(X),G)$ where

$$\pi : \mathrm{Hom}(\pi_1(X),G) \to M_B(X,G) := \mathrm{Hom}(\pi_1(X),G)/G$$

is the quotient map. This topology on $M_B(X,G)$ does not depend on the choice of presentation of the fundamental group.

Remark 4.11. A subset $S \subset M_B(X,G)$ of a quotient space can be given a topology in two ways:

(1) View S as a subset of $M_B(X,G)$ and use the subspace topology.
(2) View $\pi^{-1}(S)$ as a subset of $\mathrm{Hom}(\pi_1(X),G)$ and give this space the subspace topology, then give $S = \pi^{-1}(S)/G$ the induced quotient topology.

In general these two topologies do not agree (for example, think of $\mathbb{R}$ acting on the torus $S^1 \times S^1$ by translation by an irrational slope). However, in the case of the Betti moduli space they do.

Our goal is to decompose the Betti moduli space $M_B := M_B(X,G)$ into a well behaved part which is smooth (the 'irreducible' part) and a less well behaved part (the 'reducible' part). Before we give the definitions of irreducible and reducible representations, we recall some definitions on smoothness.

Definition 4.12. Let $U \subset \mathrm{Hom}(\pi_1(X),G)$ and $U' \subset M_B$ be open subsets; then

(1) $f : U \to \mathbb{R}$ is smooth if its the restriction of a smooth function $\tilde{f} : V \to \mathbb{R}$ on an open subset $V \subset G^{2g}$ containing U.
(2) $f' : U' \to \mathbb{R}$ is smooth if $f' \circ \pi : \pi^{-1}(U') \to \mathbb{R}$ is smooth (in the sense of part (1) of this definition).
(3) A point $p \in M_B$ is smooth of dimension n if there exists open sets $U \subset M_B$ and $V \subset \mathbb{R}^n$ and a homeomorphism $f : V \to U$ such that $p \in U$ and any $g : U \to \mathbb{R}$ is smooth if and only if $g \circ f : V \to \mathbb{R}$ is smooth.

Definition 4.13. We say

(1) a tuple $(A_1, B_1, \cdots, A_g, B_g) \in G^{2g}$ is reducible if all A_i and B_j commute pairwise (that is, $A_iB_j = B_jA_i$ and $A_iA_j = A_jA_i$ and $B_iB_j = B_jB_i$). Otherwise, we call such a tuple irreducible;

(2) a representation $\rho : \pi_1(X) \to G$ is reducible (resp. irreducible) if the corresponding tuple $(\rho(a_1), \rho(b_1), \ldots, \rho(a_g), \rho(b_g))$ is reducible (resp. irreducible).

Remark 4.14. As the condition for matrices to commute is invariant under conjugation, the notions of (ir)reducibility descend to conjugacy classes $[\rho]$ of representations. This means we can write the Betti moduli space as a disjoint union of the open set of irreducible representation M_B^{irr} and its complement, the closed subspace of reducible representations, M_B^{red}.

We recall that a maximal torus T of G is a maximal compact connected abelian subgroup. The associated Weyl group $W(T) = N(T)/T$ is a finite group which acts on T by conjugation.

Theorem 4.15. *Let X be a compact connected oriented genus g surface and G a compact connected group and let M_B^{red} denote the reducible part of the Betti moduli space of G-representations of $\pi_1(X)$. If T is a maximal torus of G, then there is a natural homeomorphism*

$$M_B^{red} \cong T^{2g}/W(T)$$

where $W(T)$ acts componentwise by conjugation.

Proof. (Idea) Given a reducible representation ρ, as the elements $\rho(a_i)$ and $\rho(b_j)$ all pairwise commute, there is a maximal torus T' containing these elements $\rho(a_i)$ and $\rho(b_j)$. Since any two maximal tori are conjugate, we can conjugate ρ and assume that $T' = T$. This defines the map $M_B^{red} \to T^{2g}$

$$[\rho] \mapsto (\rho(a_1), \rho(b_1), \ldots, \rho(a_g), \rho(b_g)).$$

The proof that this map determines the desired homeomorphism is omitted (for example, see - provide reference). ☐

Example 4.16. Let X be a compact connected oriented surface of genus 1 and G be any compact group. The fundamental group of X is abelian

$$\pi_1(X) =< a_1, b_1 : a_1b_1 = b_1a_1 >$$

and so every representation is reducible: $M_B = M_B^{\mathrm{red}} = T^2/W(T)$. In particular, for $G = SU(2)$, we have that $T \cong S^1$ and $W(T) \cong \mathbb{Z}_2$; therefore $M_B = (S^1)^2/\mathbb{Z}_2 = S^2$.

Theorem 4.17. *The reducible and irreducible part of the Betti moduli space for $G = \mathrm{SU}(2)$ and X a genus g compact connected oriented Riemann surface are described as follows:*

(1) $M_B^{red} = (S^1)^{2g}/\mathbb{Z}_2$.

(2) M_B^{irr} *is a smooth manifold of dimension 6g -6.*

To justify the second part of this theorem we need the following lemma.

Lemma 4.18. *The map $\mu_g : \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2)$ at $(A_1, B_1, \cdots, A_g, B_g)$ has rank*

- *3 if $(A_1, B_1, \ldots, A_g, B_g)$ is irreducible;*
- *0 if $A_i = \pm I$ and $B_i = \pm I$ for all $i = 1, \ldots, g$;*
- *2 otherwise.*

In particular, the restriction of μ_g to the irreducible locus $\mu_g^{irr} : (\mathrm{SU}(2)^{2g})^{irr} \to \mathrm{SU}(2)$ is a submersion.

We can now sketch the proof of the second part of the theorem. By the above lemma,

$$\mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), \mathrm{SU}(2)) = (\mu_g^{\mathrm{irr}})^{-1}(1)$$

is a smooth submanifold of $\mathrm{SU}(2)^{2g}$ of dimension $2g \dim \mathrm{SU}(2) - \dim \mathrm{SU}(2) = 6g - 3$. The conjugation action of $G = \mathrm{SU}(2)$ on this space of irreducible representations is almost free: the stabiliser of every irreducible representation is precisely $\pm I$. In particular, $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm I\}$ acts freely so the quotient M_B^{irr} is smooth with

$$\dim M_B^{\mathrm{irr}} = \dim \mathrm{Hom}^{\mathrm{irr}}(\pi_1(X), \mathrm{SU}(2)) - \dim \mathrm{SU}(2) = 6g - 6.$$

4.5. **Twisted representations of the fundamental group.** For any connection on a principal G-bundle, we can consider the holonomy around A as a group homomorphism

$$\rho_{A,p} : \{\text{Loops } \gamma : [0,1] \to X \text{ at } x\} \to G$$

which in general does not descend to give a representation of the fundamental group. Nevertheless, Atiyah and Bott prove an analogous result to Theorem 4.7 that gives a bijective correspondence between gauge equivalence classes of projectively flat connections and so-called twisted representation of the fundamental group (these are representations of a central extension of the fundamental group); for details see [1] §5 and also [9], Theorem 4.1.

We briefly discuss the case of $G = U(n)$. In this case, for any $d \in \mathbb{Z}$ (this integer corresponds to the degree), the twisted character variety (or Betti moduli space) $M_B^d(X, \mathrm{U}(n))$ is defined to be a quotient of the natural conjugation action on

$$\mathrm{Hom}^d(\hat{\pi_1}(X)_{\mathbb{R}}, \mathrm{U}(n)) := \left\{(A_1, \dots, B_g) \in \mathrm{U}(n)^{2g} : \mu_g(A_1, \dots, B_g) = \exp\left(\frac{2\pi i d}{n}\right) I_n\right\}$$

where $\mu_g : \mathrm{U}(n)^{2g} \to \mathrm{U}(n)$ is given by $\mu_g(A_1, \dots, B_g) = \prod_{i=1}^g [A_i, B_i]$. If $(n,d) = 1$ (so that $\gamma := \exp(2\pi i d/n)$ is a primitive nth root of unity), then γI_n is a regular value of μ_g and the stabiliser group of each point in the preimage $\mu^{-1}(\gamma I_n)$ is the central $U(1)$. Therefore, in the coprime case, the Betti moduli space is smooth. In general, the Betti moduli space is constructed by taking a GIT quotient that identifies some (reducible) representations. As above, the twisted character variety decomposes into a irreducible and reducible part such that the irreducible part is smooth.

5. Holomorphic structures and connections

If we want to talk about holomorphic structures on a principal bundle, then the base manifold and group should both have the structure of a complex manifold. So far we have been studying compact Lie groups and so we need to consider a complex analogue, namely the complexified group. In fact, there is a 1-1 correspondence between compact Lie groups and complex reductive groups which is given by sending a compact Lie group G to its complexification $G_{\mathbb{C}}$ and conversely by taking the maximal compact subgroup G of $G_{\mathbb{C}}$ (for example, see [16] Theorem 2.7).

Definition 5.1. Let $G_{\mathbb{C}}$ be a complex Lie group and X a complex manifold. A holomorphic structure on a principal $G_{\mathbb{C}}$-bundle $\pi : P_{\mathbb{C}} \to X$ is the structure of a complex manifold on $P_{\mathbb{C}}$ such that the transition functions are holomorphic, the $G_{\mathbb{C}}$-action on $P_{\mathbb{C}}$ is holomorphic and the projection $\pi : P_{\mathbb{C}} \to X$ is holomorphic and locally trivial in the complex topology.

Remark 5.2. If $G_{\mathbb{C}}$ is a complex linear group, then we can think of it as a complex algebraic group. Similarly, we can view our compact Riemann surface X as a smooth complex projective algebraic curve. There is an equivalence between the category of holomorphic principal $G_{\mathbb{C}}$-bundles on X (viewed as a complex manifold) and the category of complex algebraic $G_{\mathbb{C}}$-bundles on X (viewed as an algebraic curve); for details, see [21] §6, Theorem 3 or, more generally, [22].

The complex structure on $P_{\mathbb{C}}$ gives a smoothly varying complex structure on each tangent space $T_pP_{\mathbb{C}}$ (i.e. an endomorphism that squares to -1), we globally write this as an almost complex structure; that is an endomorphism $I : TP \to TP$ such that $I^2 = -1$. As this almost complex structure comes from a complex structure, it is integral (i.e. we can find local holomorphic frames).

Remark 5.3. If X is a Riemann surface, then the notions of almost complex structures and complex structures coincide: every almost complex structure is integrable (this follows from the Newlander-Nirenberg integrability criterion, alternatively, see [1] p555 for a direct proof).

Given a $G_{\mathbb{C}}$-connection $A_{\mathbb{C}}$ on a principal $G_{\mathbb{C}}$-bundle $\pi : P_{\mathbb{C}} \to X$, we have, for $p \in P$,

$$T_pP_{\mathbb{C}} = T_p^hP_{\mathbb{C}} \oplus T_p^vP_{\mathbb{C}}$$

where the spaces $T_p^h P_{\mathbb{C}} \cong T_{\pi(p)}X$ and $T_p^v P_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{C}}$ both have natural complex structures. This determines a complex structure on each tangent space $T_p P_{\mathbb{C}}$ and these complex structures fit together to give an almost complex structure on P (i.e. an endomorphism of TP that squares to -1). If we are working over a Riemann surface, this almost complex structure is integrable and so gives P the structure of a holomorphic principal $G_{\mathbb{C}}$-bundle. In higher dimensions we must impose an 'integrability condition' on our connection, in order for the associated almost complex structure to be integrable.

5.1. **Chern connections.** As above, we let X be a complex manifold and G a compact Lie group with complexification $G_{\mathbb{C}}$. We can use the complex structure on X to decompose the space of complex 1-forms on X into

$$\Omega^1(X)_{\mathbb{C}} = \Omega^{(1,0)}(X) \oplus \Omega^{(0,1)}(X)$$

where $\Omega^{(1,0)}(X)$ (resp. $\Omega^{(0,1)}(X)$) denotes the space of holomorphic (resp. anti-holomorphic) forms.

Definition 5.4. Let $P_{\mathbb{C}}$ be a principal $G_{\mathbb{C}}$-bundle on X. Then

(1) A Hermitian structure on $P_{\mathbb{C}}$ is a reduction $P \subset P_{\mathbb{C}}$ to the subgroup $G \subset G_{\mathbb{C}}$ (cf. §2.2 for details on reductions and extended principal bundles).
(2) A connection $A_{\mathbb{C}}$ on $P_{\mathbb{C}}$ is Hermitian (for $P \subset P_{\mathbb{C}}$) if it is extended from a connection on the principal G-bundle P.

The terminology 'Hermitian structure' comes from the fact that, for $G = \mathrm{U}(n)$, a choice of Hermitian structure on a $G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$-bundle corresponds to a Hermitian metric on the associated vector bundle.

Lemma 5.5. *Let I be a holomorphic structure on a principal $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}}$ and let $P \subset P_{\mathbb{C}}$ be a reduction to G; then there is a unique Hermitian connection $A_{\mathbb{C}}$ on $P_{\mathbb{C}}$ (for $P \subset P_{\mathbb{C}}$) that induces the given holomorphic structure I on $P_{\mathbb{C}}$.*

Proof. Firstly, we note that, for $p \in P$, the subspaces

$$T_p^h P := T_p P \cap I(T_p P) \tag{7}$$

are I-invariant, i.e. complex, subspaces of $T_p P_{\mathbb{C}}$. As the action is holomorphic, the infinitesimal action $\rho : P_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to TP_{\mathbb{C}}$ satisfies

$$\rho_p(i\eta) = I\rho_p(\eta) \tag{8}$$

for $p \in p$ and $\eta \in \mathfrak{g}_{\mathbb{C}}$. To prove the subspaces $T_p^h P$ define a G-connection A on P, we need to show

(1) $T_p P = T_p^v P \oplus T_p^h P$;
(2) $T_{pg}^h P = d_p R_g (T_p^h P)$.

For the first part, we note that $T_p^v P \cap T_p^h P \cong \mathfrak{g} \cap i\mathfrak{g} = 0$ and so it suffices to prove that $\dim T_p^h P = \dim X$. As $T_p P_{\mathbb{C}} = T_p P + I(T_p P)$ and

$$\dim X + \dim \mathfrak{g}_{\mathbb{C}} = T_p P_{\mathbb{C}} = \dim T_p P + \dim I(T_p P) - \dim(T_p P \cap I(T_p P)),$$

these horizontal spaces $T_p^h P = T_p P \cap I(T_p P)$ have the required dimension. The second part follows as the action of $G_{\mathbb{C}}$ on $P_{\mathbb{C}}$ is holomorphic. Hence, the horizontal subspaces defined at (7) determine a G-connection A on P.

Let $A_{\mathbb{C}}$ denote the induced Hermitian connection on $P_{\mathbb{C}}$; then $A_{\mathbb{C}}$ can be constructed as a horizontal distribution by taking

$$T_{pg}^h P_{\mathbb{C}} = d_p R_g T_p^h P$$

for $p \in P$ and $g \in G_{\mathbb{C}}$. As the action of $G_{\mathbb{C}}$ is holomorphic, these subspaces are also I-invariant. In fact, we claim that $A_{\mathbb{C}}$ is holomorphic, i.e. a $(1,0)$-form on P, and so defines the given holomorphic structure on P. Since the action is holomorphic, it follows from (8) that

$$A_{\mathbb{C}}(I\rho_p(\eta)) = A_{\mathbb{C}}(\rho_p(i\eta)) = i\eta.$$

Hence, for all vector fields ξ, we have $A_{\mathbb{C}}(I\xi) = iA_{\mathbb{C}}(\xi)$ and so $A_{\mathbb{C}} \in \Omega^{(1,0)}(P) \otimes \mathfrak{g}_{\mathbb{C}}$ as

$$A_{\mathbb{C}}(\xi + iI\xi) = A_{\mathbb{C}}(\xi) + iA_{\mathbb{C}}(I\xi) = A_{\mathbb{C}}(\xi) - A_{\mathbb{C}}(\xi) = 0.$$

To prove uniqueness, we note that if $A'_{\mathbb{C}}$ is another Hermitian connection on $P_{\mathbb{C}}$ that induces the same holomorphic structure, then the horizontal spaces H'_p defined by A' must be I-invariant. Therefore, for $p \in P$, we must have

$$H'_p \subset T_pP \cap I(T_pP) =: T_p^hP,$$

but these spaces have the same dimension and so must coincide. □

Definition 5.6. Let I be a holomorphic structure on a principal $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}}$ and let $P \subset P_{\mathbb{C}}$ be a reduction to G. Then, the unique Hermitian connection $A_{\mathbb{C}} = A_{\mathbb{C}}(I, P)$ on $P_{\mathbb{C}}$ defined above is called the Chern connection associated to I and P.

5.2. **The Atiyah-Bott isomorphism.** In this section, we restrict to the case where X is a compact Riemann surface and describe a 1-1 correspondence between holomorphic structures on a principal $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}}$ and G-connections on a given Hermitian reduction $P \subset P_{\mathbb{C}}$. For higher dimensional X, we must impose an integrability condition on our connections.

Let $\mathcal{A} = \mathcal{A}_P$ denote the space of G-connections on a fixed reduction P of $P_{\mathbb{C}}$ to $G \subset G_{\mathbb{C}}$ (equivalently, we can think of $\mathcal{A}$ as the space of Hermitian $G_{\mathbb{C}}$-connections on $P_{\mathbb{C}}$ with respect to the given Hermitian structure P). We have already seen that the space $\mathcal{A}$ is an infinite dimensional affine space modeled on $\Omega^1(X, \mathrm{ad}P)$ and that the (unitary) gauge group $\mathcal{G} = \mathrm{Aut}^G(P) = \Gamma(\mathrm{Ad}P)$ acts on $\mathcal{A}$ by pulling back connections.

We define $\mathcal{C} = \mathcal{C}_{P_{\mathbb{C}}}$ to be space of holomorphic structure on $P_{\mathbb{C}}$. Given such a holomorphic structure, we can compose the infinitesimal action $P_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to TP_{\mathbb{C}}$ with the complex structure $TP_{\mathbb{C}} \to TP_{\mathbb{C}}$. The difference of two maps obtained in this fashion is a $G_{\mathbb{C}}$-invariant horizontal $\mathfrak{g}_{\mathbb{C}}$-valued $(0,1)$-form on $P_{\mathbb{C}}$ i.e. corresponds to an $\mathrm{ad}P_{\mathbb{C}}$-valued $(0,1)$-form on X. It follows that $\mathcal{C}$ is an infinite dimensional affine space modeled on $\Omega^{(0,1)}(X, \mathrm{ad}P_{\mathbb{C}})$.

As any a Hermitian connection on $P_{\mathbb{C}}$ defines a holomorphic structure on $P_{\mathbb{C}}$, there is a natural map

$$\mathcal{A} \to \mathcal{C}$$

which is an affine linear isomorphism (as, for every holomorphic structure on $P_{\mathbb{C}}$, the Chern connection is the unique Hermitian connection inducing the same holomorphic structure). We refer to this isomorphism $\mathcal{A} \cong \mathcal{C}$ as the Atiyah-Bott isomorphism and observe that locally it corresponds to the isomorphism

$$\Omega^1(\mathfrak{g}) \cong \Omega^{(0,1)}(\mathfrak{g}_{\mathbb{C}}).$$

The complex gauge group $\mathcal{G}_{\mathbb{C}} := \mathrm{Aut}^{G_{\mathbb{C}}}(P_{\mathbb{C}}) = \Gamma(\mathrm{Ad}P_{\mathbb{C}})$ is the complexification of the unitary gauge group $\mathcal{G} = \mathrm{Aut}^G(P)$ and $\mathcal{G}_{\mathbb{C}}$ acts naturally on $\mathcal{C}$ by pulling back complex structures along complex gauge transformations. The complex gauge group orbits correspond to isomorphism classes of holomorphic structures. We can artificially extend the $\mathcal{G}$-action on $\mathcal{A}$ to a $\mathcal{G}_{\mathbb{C}}$-action on $\mathcal{A}$ via the Atiyah–Bott isomorphism $\mathcal{A} \cong \mathcal{C}$. Hence, for $\Phi \in \mathcal{G}_{\mathbb{C}}$ and a connection A on P, we define the action to be

$$\Phi \cdot A = A(\Phi^*(I_{A_{\mathbb{C}}}), P);$$

that is, it is the Chern connection associated to the Hermitian structure P and the pullback along Φ of the holomorphic structure on $P_{\mathbb{C}}$ defined by $A_{\mathbb{C}}$.

Remark 5.7. We note that in order to define the Atiyah-Bott isomorphism $\mathcal{C} \cong \mathcal{A}$, it was necessary to fix a choice of Hermitian structure $P \subset P_{\mathbb{C}}$ (in particular, to construct the Chern connection associated to a holomorphic structure we need a Hermitian structure). If P and P' are two different Hermitian structures on $P_{\mathbb{C}}$; then they are related by a complex gauge transformation Φ (i.e. $\Phi^*P = P'$). A connection $A_{\mathbb{C}}$ on $P_{\mathbb{C}}$ is Hermitian for P if and only if $A_{\mathbb{C}} \cdot \Phi$ is Hermitian for P'. Therefore, if we consider $\mathcal{G}_{\mathbb{C}}$-orbits, the choice of Hermitian structure no longer plays a role. In fact, the space of Hermitian structures on $P_{\mathbb{C}}$ can naturally be identified with $\mathcal{G}_{\mathbb{C}}/\mathcal{G}$.

5.3. **Hitchin–Kobayashi.** The goal of this section is to reinterpret the moduli space of gauge equivalence classes of projectively flat connections on P as a moduli space of certain 'polystable' holomorphic structures on $P_{\mathbb{C}}$. This equivalence in full generality is known as the Hitchin–Kobayashi correspondence. It can be seen as a generalisation of the Narasimhan–Seshadri correspondence [13] for $G = \mathrm{U}(n)$; we study this special case in much greater detail in Section 6 below. For $G = \mathrm{U}(n)$, this correspondence was generalised to higher dimensional bases by Kobayashi [6] and Donaldson [4]. Later Ramanan and Ramanathan studied principal G-bundles for compact connected groups G (cf. [18, 19]) and the analogous correspondence for was proved by Ramanathan and Submramanian [20]. We first make the following definitions.

Definition 5.8. A connection $A_{\mathbb{C}}$ on a holomorphic principal $G_{\mathbb{C}}$-bundle $P_{\mathbb{C}}$ is Hermitian–Einstein if

(1) there is a reduction $P \subset P_{\mathbb{C}}$ to $G \subset G_{\mathbb{C}}$ such that $A_{\mathbb{C}}$ is Hermitian and the holomorphic structure defined by this connection agrees with the given holomorphic structure on $P_{\mathbb{C}}$;
(2) the connection A on P is projectively flat i.e. $\star F_A \in \Omega^0(X, \mathrm{ad}P) = \mathrm{Hom}^G(P, \mathfrak{g})$ is a constant element in the centre of $\mathfrak{g}$.

Definition 5.9. A holomorphic principal $G_{\mathbb{C}}$-bundle $\pi : P_{\mathbb{C}} \to X$ is (semi)stable if for every holomorphic reduction P_Q of $P_{\mathbb{C}}$ to a maximal parabolic subgroup $Q \subset G_{\mathbb{C}}$ and every dominant character $\chi : Q \to \mathbb{C}^*$ we have

$$\deg P_Q(\chi) < 0 \text{ (for stability) or } \deg P_Q(\chi) \leq 0 \text{ (for semistability)}$$

where $P_Q(\chi) = P_Q \times_\chi \mathbb{C}$ is the line bundle associated to the character χ and Q-bundle P_Q. We say $P_{\mathbb{C}}$ is polystable if it has a holomorphic reduction P_L to a Levi subgroup L of a parabolic group $Q \subset G$ such that P_L is stable and $\deg P_L(\chi) = 0$ for any dominant character $\chi : L \to \mathbb{C}^*$ which is trivial on centre of G.

Theorem 5.10. *(Hitchin–Kobayashi correspondence) A holomorphic principal $G_{\mathbb{C}}$-bundle $\pi : P \to X$ is polystable if and only if it admits a Hermitian–Einstein connection. Moreover, this connection is unique.*

This gives the so-called non-abelian Hodge correspondence: a diffeomorphism between the gauge theoretic 'de Rham' moduli space $\mathcal{M}_{dR}$ of Hermitian-Einstein $G_{\mathbb{C}}$-connections on a principal $G_{\mathbb{C}}$-bundle and the algebraic 'Dolbeault' moduli space $\mathcal{M}_{Dol}$ of polystable algebraic (or, equivalently, holomorphic) principal $G_{\mathbb{C}}$-bundles.

In the next section, we give the proof of this correspondence in the case $G = \mathrm{U}(n)$; where the correspondence is referred to as the Narasimhan–Seshadri correspondence. In this case, a principal $G_{\mathbb{C}}$-bundle corresponds to a rank n complex vector bundle E and a reduction to G is just a Hermitian metric on E.

6. The case of $\mathrm{U}(n)$: The Narasimhan–Seshadri correspondence

In this section, we consider the group $G = \mathrm{U}(n)$ and its complexification $G_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. The category of principal $\mathrm{GL}_n(\mathbb{C})$-bundles on a compact Riemann surface X is equivalent to the category a rank n complex vector bundles on X. Furthermore a reduction of a principal $\mathrm{GL}_n(\mathbb{C})$-bundle to $\mathrm{U}(n)$ is equivalent to a Hermitian metric on the corresponding vector bundle. This means we can view $\mathcal{A}$ as a space of 'unitary (affine) connections' on a rank n Hermitian vector bundle and the space $\mathcal{C}$ as the space of holomorphic structures on a rank n complex vector bundle. In this case, the Hitchin-Kobayashi correspondence is known as the Narasimhan–Seshadri correspondence; it relates projectively flat unitary connections with polystable holomorphic vector bundles. The original formulation of the Narasimhan–Seshadri correspondence related polystable holomorphic vector bundles with (twisted) unitary representations of the fundamental group of X [13]. Donaldson later reformulated this as a correspondence relating Hermitian–Einstein connections and polystable holomorphic vector bundles [3]. Our proof will follow the lines of Donaldson's proof in which Uhlenbeck's compactness result [26] plays a vital role.

6.1. **Vector bundles and principal $\mathrm{GL}_n(\mathbb{C})$-bundles.** There is an equivalence of categories between the category of principal $\mathrm{GL}_n(\mathbb{C})$-bundles on X and the category a rank n complex vector bundles on X. The equivalence works by sending a principal $\mathrm{GL}_n(\mathbb{C})$-bundle $P_{\mathbb{C}}$ to the vector bundle

$$E := P_{\mathbb{C}} \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n$$

associated to the standard representation of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$. The inverse construction is given by taking the frame bundle of E.

If we have a Hermitian structure i.e. a reduction $P \subset P_{\mathbb{C}}$ to $G \subset G_{\mathbb{C}}$, then this correspondences to a Hermitian metric h on $E = P_{\mathbb{C}} \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n$. In fact, there is an equivalence of categories between the category of principal $\mathrm{U}(n)$-bundles on X and the category a rank n Hermitian vector bundles on X.

In terms of vector bundles, the gauge group $\mathcal{G} = \mathrm{Aut}^G(P) = \Gamma(\mathrm{Ad}P)$ and its complexification $\mathcal{G}_{\mathbb{C}} = \mathrm{Aut}^{G_{\mathbb{C}}}(P_{\mathbb{C}}) = \Gamma(\mathrm{Ad}P_{\mathbb{C}})$ can be interpreted as the group $\mathcal{G} = \mathrm{U}(E,h)$ of unitary automorphisms of (E,h) and the automorphism group $\mathcal{G}_{\mathbb{C}} = \mathrm{Aut}(E)$. The adjoint bundle

$$\mathrm{ad}P_{\mathbb{C}} = P \times_{\mathrm{GL}_n(\mathbb{C})} \mathfrak{gl}_n(\mathbb{C}) \quad (\text{resp. } \mathrm{ad}P = P \times_{U(n)} \mathfrak{u}(n))$$

is naturally isomorphic to the vector bundle $\mathrm{End}(E)$ (resp. $\mathfrak{u}(E)$, the bundle of skew-Hermitian endomorphisms).

6.2. **Unitary connections.** Let (E,h) be a smooth Hermitian vector bundle of rank n over X. We let $P_{\mathbb{C}}$ (resp. P) denote the associated principal $\mathrm{GL}_n(\mathbb{C})$-bundle (resp. $\mathrm{U}(n)$-bundle) of frames (resp. h-unitary frames) of E; then $E = P_{\mathbb{C}} \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n$ is the vector bundle associated to the standard representation of $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$.

Let $A_{\mathbb{C}}$ be a connection on $P_{\mathbb{C}}$ and $d_{A_{\mathbb{C}}} : \Omega^0(X,E) \to \Omega^1(X,E)$ be the corresponding affine connection (cf. §2.1). Given a local frame $s_1, \dots, s_n$ of E over U_α, we defined the associated local matrix of 1-forms $d_{A_{\mathbb{C}},\alpha} = \{(d_{A_{\mathbb{C}},\alpha})_{ij}\}_{i,j} \in \Omega^1(U_\alpha) \otimes \mathfrak{gl}_n(\mathbb{C})$ by $d_{A_{\mathbb{C}}}(s_i) = \sum_j (d_{A_{\mathbb{C}},\alpha})_{ij} s_j$. If $A_{\mathbb{C}}$ is a Hermitian connection (that is, it is extended from a connection A on a reduction $P \subset P_{\mathbb{C}}$ to $G \subset G_{\mathbb{C}}$), then the associated affine connection $d_A : \Omega^0(X,E) \to \Omega^1(X,E)$ is a unitary connection on (E,h) in the following sense.

Definition 6.1. A connection $\nabla : \Omega^0(X,E) \to \Omega^1(X,E)$ on a Hermitian vector bundle (E,h) is unitary (for h) if for any h-unitary local frame $s_1, \dots, s_n$ of E over U_α the associated local matrix of 1-forms $\nabla_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{gl}_n(\mathbb{C})$ is skew-Hermitian; that is, $\nabla_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{u}(n)$. Equivalently, ∇ is unitary if, for all sections s_i of E, we have

$$dh(s_1, s_2) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2)).$$

Let $F_{A_{\mathbb{C}}} \in \Omega^2(X, \mathrm{ad}P_{\mathbb{C}})$ denote the associated curvature form of the connection $A_{\mathbb{C}}$ on $P_{\mathbb{C}}$; then we can view this as corresponding to the curvature form $F_{A_{\mathbb{C}}} \in \Omega^2(X, \mathrm{End}E)$ associated to the corresponding affine connection $d_{A_{\mathbb{C}}}$. The map

$$F_A := d_A \circ d_A : \Omega^0(X,E) \to \Omega^2(X,E)$$

is $\mathcal{C}^\infty(X)$-linear (due to the Leibniz rule) and so defines a global 2-form $F_A \in \Omega^2(X, \mathrm{End}E)$. If d_A is a unitary connection on (E,h), then it has curvature $F_A \in \Omega^2(X, \mathfrak{u}(E))$.

Following the above discussion, we can think of the space $\mathcal{A}$ of $\mathrm{U}(n)$-connections on P as the space of unitary connections on (E,h). If we take the second point of view, then $\mathcal{A}$ is an affine space modeled on $\Omega^1(X, \mathfrak{u}(E))$. The unitary gauge group $\mathcal{G} = \mathcal{G}_{(E,h)} = \mathrm{U}(E,h)$ is the group of unitary automorphisms of (E,h). Moreover, $\mathrm{Lie}\,\mathcal{G} \cong \Omega^0(X, \mathfrak{u}(E))$ and $\mathrm{Lie}\,\mathcal{G}^* \cong \Omega^0(X, \mathfrak{u}(E))^* \cong \Omega^2(X, \mathfrak{u}(E))$.

We fix a metric g on our Riemann surface X and let $\star : \Omega^2(X) \to \Omega^0(X)$ be the associated Hodge star operator.

Definition 6.2. An affine connection $\nabla : \Omega^0(X,E) \to \Omega^1(X,E)$ on a complex vector bundle E is Hermitian–Einstein if

(1) ∇ is unitary for some hermitian metric h on E;
(2) ∇ is projectively flat, i.e. $\star F_\nabla \in \Omega^0(X, \mathfrak{u}(E,h))$ is a constant imaginary scalar multiple of the identity map I_E on E.

6.3. **Holomorphic structures.** Let $\pi : E \to X$ be a complex vector bundle.

Definition 6.3. A Dolbeault operator on E is a $\mathbb{C}$-linear map $\overline{\partial} : \Omega^0(X, E) \to \Omega^{(0,1)}(X, E)$ satisfying the Leibniz rule.

Lemma 6.4. *For a compact Riemann surface X, a holomorphic structure on a vector bundle E over X is equivalent to a Dolbeault operator on E.*

Proof. Given a holomorphic structure on E, we can define a Dolbeault operator by taking local holomorphic frames $s_1, \dots s_n$ and defining $\overline{\partial}(s_i) := 0$ and then use the Leibniz rule and linearity to extend:

$$\overline{\partial}(\sum_{i=1}^{n} f_i s_i) = \sum_{i=1}^{n} \overline{\partial} f_i \otimes s_i.$$

Conversely, given a Dolbeault operator $\overline{\partial}$ we want to find local holomorphic frames; that is, $s_1, \dots, s_n$ such that $\overline{\partial}(s_i) = 0$. We start with a local smooth frame $\sigma_1, \dots, \sigma_n$ and aim to find a $n \times n$ matrix of functions $f = (f_{ij})$ on X such that $s = f\sigma$ gives a holomorphic frame. We write $\overline{\partial}(s) = \theta s$ for a matrix $\theta = \theta_{ij}$ of 1-forms; then solving $\overline{\partial}(f\sigma) = 0$ is equivalent to $\overline{\partial} f + f\theta = 0$. As we are working over a Riemann surface, there is no integrability condition and we can solve for f (cf. [1] p555). □

The space $\mathcal{C}$ of holomorphic structures on $P_{\mathbb{C}}$ can then be thought of as the space of holomorphic structures on the corresponding complex vector bundle E or, equivalently as the space of Dolbeault operators on E. Viewed as the space of Dolbeault operators on E, the space $\mathcal{A}$ is an infinite dimensional affine space modeled on $\Omega^{(0,1)}(X, \mathrm{End}E)$. The Atiyah-Bott isomorphism $\mathcal{A} \cong \mathcal{C}$ in this case can be viewed as sending a unitary connection d_A to the Dolbeault operator given by its $(0, 1)$-part (it is an isomorphism as the Chern connection is the unique unitary connection associated to a given holomorphic structure and Hermitian metric h).

The complex gauge group $\mathcal{G}_{\mathbb{C}} = \mathrm{Aut}(E)$ acts by pulling back Dolbeault operators (or equivalently holomorphic structures). We can transfer the action of $\mathcal{G}_{\mathbb{C}} = \mathrm{Aut}(E)$ on $\mathcal{C}$ to an action on $\mathcal{A}$ by this isomorphism and this extends the previously defined action of $\mathcal{G} = \mathrm{U}(E, h)$ to the complexification $\mathcal{G}_{\mathbb{C}}$.

Remark 6.5. Again we note that the isomorphism $\mathcal{A} \cong \mathcal{C}$ depends on the choice of Hermitian metric h. However, we can identify the space of Hermitian metrics on E with $\mathrm{Aut}E/\mathrm{U}(E, h) = \mathcal{G}_{\mathbb{C}}/\mathcal{G}$. Any two Hermitian metrics h and h' on E can be related by a complex gauge transformation; that is, $h = \Phi^* h'$ for a complex gauge transformation $\Phi : E \to E$. Then we claim that d_A is a h'-unitary connection on E if and only if $d_A \cdot \Phi$ is h-unitary. For example, suppose d_A is h'-unitary, then we check that $\Phi \cdot d_A$ is h-unitary:

$$\begin{aligned} dh(s_1, s_2) &= dh'(\Phi_* s_1, \Phi_* s_2) \\ &= h'(d_A(\Phi_* s_1), \Phi_* s_2)) + h'(\Phi_* s_1, d_A(\Phi_* s_2)) \\ &= h((\Phi \cdot d_A) s_1, s_2) + h(s_1, (\Phi \cdot d_A) s_2) \end{aligned}$$

Hence, if we consider $\mathcal{G}_{\mathbb{C}}$-orbits, then the choice of Hermitian metric is no longer important.

6.4. **Topological invariants of vector bundles.** Let X be a compact Riemann surface and we choose a Riemannian metric g whose volume form ω induces the given orientation on X. Topologically, vector bundles on X are classified by their rank and degree:

$$\deg(E) := \int_X c_1(E).$$

From the Chern-Weil description of characteristic classes, we can define the first chern class using a unitary connection d_A on E. More precisely, we take the curvature F_A of this connection and define

$$c_1(E) := \frac{i}{2\pi} \mathrm{Tr}(F_A) \in H^2(X, \mathbb{R}).$$

Hence, if E admits a flat connection, it has degree zero. More generally, for a projectively flat connection F_A with $\star F_A = -i\mu I_E$, we have

$$\deg(E) = \int_X \frac{i}{2\pi}\mathrm{Tr}(F_A) = \int_X \frac{i}{2\pi}\mathrm{Tr}(\star F_A)\omega = \int_X \frac{1}{2\pi}\mathrm{Tr}(\mu I)\omega = \frac{\mu \mathrm{rk} E}{2\pi}\int_X \omega.$$

If we normalise our metric on X so that $\int_X \omega = 2\pi$, then

$$\mu = \frac{\deg(E)}{\mathrm{rk}(E)}.$$

This quantity is called the slope of the vector bundle E and naturally appears in the algebraic notion of semistability.

6.5. **Semistability.** In order to construct moduli spaces of algebraic vector bundles over smooth complex projective curves, Mumford introduced a notion of semistability for algebraic vector bundles. This notion naturally arose from his work on geometric invariant theory (GIT) [10]. In keeping with the rest of this section, we state all notions for holomorphic bundles; however, if one is interested in the algebraic notions, one can simply replace the word holomorphic by algebraic. The slope of a complex vector bundle E on X is the ratio

$$\mu(E) := \frac{\deg E}{\mathrm{rk}\, E}.$$

Remark 6.6. The degree and rank are both additive on short exact sequences of bundles

$$0 \to E \to F \to G \to 0.$$

It then follows that we have:

(1) If two out of the three bundles have the same slope μ, the third also has slope μ;
(2) $\mu(E) < \mu(F)$ (resp. $\mu(E) > \mu(F)$) if and only if $\mu(F) < \mu(G)$ (resp. $\mu(F) > \mu(G)$).

To distinguish between smooth and holomorphic bundles, we write $\mathcal{E}$ for a holomorphic vector bundle and E for the underlying smooth bundle.

Definition 6.7. A holomorphic vector bundle $\mathcal{E}$ is stable (resp. semistable) if every proper non-zero holomorphic subbundle $\mathcal{S} \subset \mathcal{E}$ satisfies

$$\mu(\mathcal{S}) < \mu(\mathcal{E}) \qquad \text{(resp. } \mu(\mathcal{S}) \leq \mu(\mathcal{E}) \text{ for semistability).}$$

A holomorphic vector bundle $\mathcal{E}$ is polystable if it is a direct sum of stable bundles of the same slope.

Lemma 6.8. *Let $\mathcal{L}$ be a holomorphic line bundle and $\mathcal{E}$ a holomorphic vector bundle; then*

(1) *$\mathcal{L}$ is stable.*
(2) *If $\mathcal{E}$ is stable (resp. semistable), then $\mathcal{E} \otimes \mathcal{L}$ is stable (resp. semistable).*

Proof. As $\mathcal{L}$ has rank 1, it has no proper non-zero subbundles so is trivially stable. For the second statement, we note that if $\mathcal{S} \subset \mathcal{E} \otimes \mathcal{L}$, then $\mathcal{S} \otimes \mathcal{L}^\vee \subset \mathcal{E}$. Then as $\deg \mathcal{S} \otimes \mathcal{L}^\vee = \deg \mathcal{S} - \mathrm{rk}\, \mathcal{S} \deg \mathcal{L}$ and $\mathcal{E}$ is stable, we have

$$\mu(\mathcal{S} \otimes \mathcal{L}^\vee) < \mu(\mathcal{E}) \Rightarrow \mu(\mathcal{S}) < \mu(\mathcal{E}) + \deg \mathcal{L} = \mu(\mathcal{E} \otimes \mathcal{L})$$

and analogously for semistability. □

Remark 6.9. Given a map of holomorphic vector bundles $f : \mathcal{E} \to \mathcal{G}$, the kernel (resp. image) is not necessarily a subbundle of $\mathcal{E}$ (resp. $\mathcal{G}$) as the quotient $\mathcal{E}/\mathrm{ker} f$ may not exist as a vector bundle. However, we can construct a subbundle $\mathcal{K} \subset \mathcal{E}$ (resp. $\mathcal{I} \subset \mathcal{F}$) that is 'generically generated' by the kernel (resp. image) such that

$$\mathrm{rk}\, \mathcal{K} = \mathrm{rk}\, \mathrm{ker} f \quad (\text{resp. } \mathrm{rk}\, \mathcal{I} = \mathrm{rk}\, \mathrm{Im} f) \quad \text{and} \quad \deg \mathcal{K} \geq \deg \mathrm{ker} f \quad (\text{resp. } \deg \mathcal{I} \geq \deg \mathrm{Im} f).$$

More precisely, in sheaf theoretic terms, we can view the kernel $\mathrm{ker} f$ as a subsheaf of the locally free sheaf (i.e. vector bundle) $\mathcal{E}$ and then we define $\mathcal{K}$ to be the inverse image in $\mathcal{E}$ of the torsion subsheaf of $\mathcal{E}/\mathrm{ker} f$. Then $\mathcal{K}$ is torsion free and also $\mathcal{F}/\mathcal{K}$ is torsion free; i.e. they are both vector bundles, as we are working over a base of complex dimension 1 and so torsion free sheaves are also locally free.

Lemma 6.10. *Let $f : \mathcal{E} \to \mathcal{F}$ be a non-zero map of vector bundles; then*

(1) *If $\mathcal{E}$ and $\mathcal{F}$ are semistable, $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$.*
(2) *If $\mathcal{E}$ and $\mathcal{F}$ are stable of the same slope, then f is an isomorphism.*

In particular, every stable bundle is simple i.e. $\operatorname{End}\mathcal{E} = \mathbb{C}$.

Proof. If $f : \mathcal{E} \to \mathcal{F}$ is non-zero, then let $\mathcal{K} \subset \mathcal{E}$ and $\mathcal{I} \subset \mathcal{F}$ be the subbundles generically generated by the kernel and image of f. We have that

$$\operatorname{rk}\mathcal{E} = \operatorname{rk}\mathcal{K} + \operatorname{rk}\mathcal{I} \quad \text{and} \quad \deg\mathcal{E} \leq \deg\mathcal{K} + \deg\mathcal{I}.$$

By semistability, we have that $\mu(\mathcal{K}) \leq \mu(\mathcal{E})$ and similarly $\mu(\mathcal{I}) \leq \mu(\mathcal{F})$. Then, we have

$$\mu(\mathcal{E}) \leq \mu(\mathcal{E}/\mathcal{K}) \leq \mu(\mathcal{I}) \leq \mu(\mathcal{F}).$$

If both vector bundles are stable of the same slope, then we have $\mu(\mathcal{K}) < \mu(\mathcal{E})$ if $\mathcal{K} \neq 0$ and $\mu(\mathcal{I}) < \mu(\mathcal{F})$ if $\mathcal{I} \neq \mathcal{F}$. However, if either $\mathcal{K} \neq 0$ or $\mathcal{I} \neq \mathcal{F}$, then we get a contradiction:

$$\deg\mathcal{E} \leq \deg\mathcal{K} + \deg\mathcal{I} < \operatorname{rk}\mathcal{K}\mu(\mathcal{E}) + \operatorname{rk}\mathcal{I}\mu(\mathcal{F}) = \operatorname{rk}\mathcal{E}\mu(\mathcal{E}) = \deg\mathcal{E}.$$

Hence f is an isomorphism.

Finally, we prove that any non-zero endomorphism $f : \mathcal{E} \to \mathcal{E}$ of a stable vector bundle is given by scalar multiplication by an element in $\mathbb{C}^*$. From above, we already know that f is an isomorphism. If we take any $x \in X$ and any eigenvalue λ of the linear map $f_x : E_x \to E_x$. Since $f - \lambda I$ is no longer an isomorphism, it must be zero. Hence, $\operatorname{End}\mathcal{E} = \mathbb{C}$. ☐

6.6. **The statement.** Let E be a rank n complex vector bundle. We recall that if d_A is a Hermitian–Einstein (i.e. projectively flat unitary) connection on E, then $\star F_A = -i\mu I_E$ and by Chern-Weil theory, we have that $\mu = \mu(E)$ is the slope of this vector bundle (cf. Example 3.5 and §6.4). Under the Atiyah-Bott isomorphism between the space of unitary connections on E (for a fixed Hermitian metric h) and the space of holomorphic vector bundle structures on E, the Narasimhan–Seshadri correspondence describes which holomorphic bundle structures correspond to Hermitian–Einstein connections.

Theorem 6.11 (Narasimhan-Seshadri-Donaldson [13, 3])**.** *A holomorphic vector bundle $\mathcal{E}$ is polystable if and only if it admits a Hermitian–Einstein connection. Moreover, this connection is unique.*

One direction ('$\Leftarrow$') follows by carefully describing holomorphic subbundles of $\mathcal{E}$. More precisely, for any holomorphic subbundle $\mathcal{S}$ of a holomorphic bundle $\mathcal{E}$, we need to show that

$$\mu(\mathcal{S}) \leq \mu(\mathcal{E})$$

with equality if and only if $\mathcal{S}$ is a direct summand on $\mathcal{E}$.

A holomorphic subbundle $\mathcal{E} \subset \mathcal{S}$ determines a short exact sequence of holomorphic bundles

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$$

and we write the underlying smooth bundles as

$$0 \to S \to E \to Q \to 0.$$

In the smooth category, every such short exact sequence splits and so we choose a smooth splitting $E = S \oplus Q$. We want to use the Hermitian–Einstein connection d_A on E or, more precisely, the corresponding dolbeault operator $\overline{\partial}$ on E that gives the given holomorphic structure $\mathcal{E}$, to prove $\mathcal{E}$ is polystable by bounding the degree of S. The idea is to use the Chern-Weil description of the degree. We can express the Dolbeault operator $\overline{\partial}$ as a 2×2 block matrix with respect to the decomposition $E = S \oplus Q$ and, as $\mathcal{S}$ is a holomorphic subbundle, this matrix is block upper triangular:

$$\overline{\partial} = \begin{pmatrix} \overline{\partial}_S & \beta \\ 0 & \overline{\partial}_Q \end{pmatrix}$$

where $\beta \in \Omega^{(0,1)}(X, \operatorname{Hom}(Q, S))$. The form β is the obstruction to the short exact sequence of holomorphic bundles splitting; that is, $\beta = 0$ if and only if we have a holomorphic splitting $\mathcal{E} = \mathcal{S} \oplus \mathcal{Q}$. Let $\pi_S : E \to E$ be the projection onto S.

Remark 6.12. In fact, as we have a Hermitian metric h on E we can take an orthogonal splitting and let $\pi_S : E \to E$ be the orthogonal projection onto S; then $\pi_S^2 = \pi_S$ and $\pi_S^* = \pi_S$. As $\mathcal{S} \subset \mathcal{E}$ is a holomorphic subbundle, the projection π_S satisfies $\overline{\partial}\pi_S = \pi_S\overline{\partial}\pi_S$. Then holomorphic subbundles $\mathcal{S}$ of $\mathcal{E}$ are characterised by endomorphisms $\pi : E \to E$ satisfying the conditions:

(1) $\pi^2 = \pi$;
(2) $\pi^* = \pi$;
(3) $\overline{\partial}\pi = \pi\overline{\partial}\pi$.

Let d_A denote the unitary Chern connection on E associated to this holomorphic structure and the Hermitian metric h; then the Chern connection has block skew-Hermitian form with respect to the smooth splitting

$$d_A = \begin{pmatrix} d_{A_S} & \beta \\ -\beta^* & d_{A_Q} \end{pmatrix}$$

and the associated curvature 2-form is given by

$$F_A = \begin{pmatrix} F_{A_S} - \beta \wedge \beta^* & d\beta \\ -d\beta^* & F_{A_Q} - \beta^* \wedge \beta \end{pmatrix}.$$

With this notation in place, we can prove one direction of the Narasimhan–Seshadri correspondence.

Proposition 6.13. *If a holomorphic vector bundle $\mathcal{E}$ admits a Hermitian–Einstein connection d_A, then $\mathcal{E}$ is polystable.*

Proof. If $\mathcal{S}$ is a holomorphic subbundle of a holomorphic bundle $\mathcal{E}$ and π_S denotes the corresponding orthogonal projection, then as we saw above $F_{A_S} = \pi_S F_A \pi_S + \beta \wedge \beta^*$. Now we can use this curvature to calculate the degree of S:

$$\deg S = \frac{1}{2\pi}\int_X \operatorname{tr}(i \star F_{A_S})\omega = \frac{i}{2\pi}\int_X \operatorname{tr}(\pi_S \star F_A \pi_S)\omega + \frac{i}{2\pi}\int_X \operatorname{tr}(\star(\beta \wedge \beta^*)).$$

As $2\pi i|\beta|^2 = \operatorname{tr}(\star(\beta \wedge \beta^*))$, we have that

$$\deg S \leq \frac{1}{2\pi}\int_X \operatorname{tr}(i\pi_S \star F_A \pi_S)\omega$$

with equality if and only if $\beta = 0$ i.e. we have a holomorphic splitting $\mathcal{E} = \mathcal{S} \oplus \mathcal{Q}$. As d_A is Hermitian–Einstein, i.e. $\star F_A = -i\mu I_E$ and $\mu = \mu(E)$, we have

$$\deg S \leq \frac{1}{2\pi}\int_X \operatorname{tr}(\pi_S \mu(E) I_E \pi_S)\omega = \frac{1}{2\pi}\int_X \mu(E)\operatorname{rk}(S)\omega = \mu(E)\operatorname{rk}(S)$$

with equality if and only if $\mathcal{S}$ is a direct summand of $\mathcal{E}$. In the case of equality, there is a holomorphic splitting $\mathcal{E} = \mathcal{S} \oplus \mathcal{Q}$ and all three bundle have the same slope. We can then repeat the argument with the direct summands. In this process the rank decreases and as every rank one vector bundle is stable, we see that $\mathcal{E}$ decomposes as a direct sum of stable vector bundles of the same slope. □

6.7. **Donaldson's proof of the Narasimhan–Seshadri correspondence.** It remains to prove the implication '⇒' in the Narasimhan–Seshadri correspondence; that is, that every polystable holomorphic vector bundle $\mathcal{E}$ admits a unique Hermitian–Einstein connection. In this section, we sketch the main steps of Donaldson's proof [3] (our exposition follows closely that given in [27] for the analogous statement involving Higgs bundles).

Step 1. Choose any unitary connection $d_{A_0} \in \mathcal{A}$ in the complex gauge orbit $\mathcal{O} = \mathcal{G}_{\mathbb{C}} \cdot \mathcal{E}$ corresponding to the holomorphic structure $\mathcal{E}$. The Yang-Mills function

$$S_{YM} : \mathcal{A} \to \mathbb{R} \qquad d_A \mapsto ||F_A||^2 = \frac{1}{2}\int_X \operatorname{Tr}(F_A \wedge \star F_A)$$

can be viewed as a Morse function on $\mathcal{A}$ and so we take a minimising sequence d_{A_i} for the Yang-Mills functional starting at d_{A_0}. As the Yang-Mills flow is contained in the complex gauge group orbits, there are complex gauge transformations g_i such that $d_{A_0} = g_i \cdot d_{A_i}$.

Step 2. The next step uses Uhlenbeck compactness [26] to prove there is a weakly convergent subsequence in this orbit. More precisely, Uhlenbeck compactness says, for a sequence d_{A_j} of L^2_1-connections such that $||F_{A_j}||_{L^2}$ is uniformly bounded, there exits a subsequence j_n and gauge transformations $g_{j_n} \in L^2_1$ such that $g_{j_n}(d_{A_{j_n}}) \to d_{A_\infty}$ weakly in L^2_1. To apply this result, Donaldson shows the L^2-norms of curvatures of the connections in the minimising sequence are uniformly bounded. Then, we can write $d_{A_\infty} = d_{A_{j_n}} + \epsilon_{j_n}$ for $\epsilon_{j_n} \to 0$ weakly in L^2_1 and, by the Sobolev embedding theorem, we can assume that $a_j \to 0$ in some C^k.

Step 3. The limit d_{A_∞} lies in the closure of the complex gauge orbit $\mathcal{O}$ and is critical for the Yang-Mills functional; therefore, it is projectively flat. To complete the proof, it remains to prove that this limit lies in the orbit $\mathcal{O} = \mathcal{G}_{\mathbb{C}} \cdot \mathcal{E}$. As the limit d_{A_∞} is Hermitian–Einstein, it corresponds to a polystable vector bundle $\mathcal{E}_\infty$ by Proposition 6.13. In addition, we claim that there is a non-zero holomorphic map $g_\infty : \mathcal{E} \to \mathcal{E}_\infty$ obtained as a limit of the g_{j_n}. To see this, we note that $||g_{j_n}||_{C^k} \leq C||g_{j_n}||_{L_2}$ and so if we rescale the gauge transformations g_{j_n} to all have L^2-norm equal to 1, then $g_{j_n} \to g_\infty$ in C^k by compactness. Furthermore, the continuous map g_∞ cannot be identically zero as g_{j_n} all have norm 1. Finally, to conclude that $g_\infty : \mathcal{E} \to \mathcal{E}_\infty$ is holomorphic, we prove that $\overline{\partial}_{A_\infty}(g_\infty) \to 0$ weakly (using the fact that g_{j_n} are holomorphic) and then apply Weyl's lemma.

Step 4. Under the Yang-Mills flow the slope of a vector bundle can only decrease; therefore, $\mu(\mathcal{E}_\infty) \leq \mu(\mathcal{E})$. Next, we use the polystability of $\mathcal{E}$: we have a holomorphic non-zero map $g_\infty : \mathcal{E} \to \mathcal{E}_\infty$ of polystable vector bundles and so $\mu(\mathcal{E}) \leq \mu(\mathcal{E}_\infty)$ by Lemma 6.10. Hence $\mathcal{E}$ and $\mathcal{E}_\infty$ have the same slope. Since a non-zero map of stable bundles of the same slope is an isomorphism, we see that f is an isomorphism. Therefore $\mathcal{E}_\infty \in \mathcal{O}$ and this completes the proof.

7. Moduli of algebraic vector bundles

In this section, we describe the algebraic construction of the moduli space of (semi)stable vector bundles on a smooth complex projective algebraic curve X. As is often the case, this moduli space is constructed using geometric invariant theory.

We briefly outline the idea of the construction. First, we find a variety S whose points correspond to semistable vector bundles with fixed invariants (in the terminology introduced below, this is a family of semistable bundles parametrised by S) and this family locally encodes the information about all other such families of bundles (it is a local universal family in the terminology below). In addition, we show there is a reductive algebraic group G acting on S such that two points lie in the same orbits if and only if they correspond to isomorphic bundles. Then the moduli space is constructed as a quotient $S//G$ of this action using the techniques of geometric invariant theory. In fact, the notion of semistability for vector bundles was introduced by David Mumford following his study of semistability in geometric invariant theory. The construction of the moduli space of stable vector bundles on a Riemann surface was first given by Seshadri [23]; we will follow closely the construction given by Newstead in [14] (see also [15]).

In the sections below, we give an overview of moduli spaces in algebraic geometry and their construction from GIT; for this, the Tata institute lecture notes of Peter Newstead [15] are still one of the best references available. We then describe how to construct a local universal family of semistable sheaves with fixed invariants and then construct the associated quotient. Finally, we describe this quotient as a moduli space. We start with a very quick survey of algebraic vector bundles over a smooth projective curve.

7.1. Algebraic vector bundles.

In this section, we state some properties of algebraic vector bundles over a smooth complex projective algebraic curve X that will be needed below. We will often make use of the equivalence between the category of algebraic vector bundles on X

and the category of locally free sheaves. We recall that this equivalence is given by associating to an algebraic vector bundle F the sheaf $\mathcal{F}$ of sections of F. Under this equivalence, the trivial line bundle $X \times \mathbb{C}$ on X corresponds to the 'structure sheaf' $\mathcal{O}_X$ whose sections are simply functions on (open sets of) X. Similarly, the rank m trivial bundle corresponds to the free sheaf $\mathcal{O}_X^{\oplus m}$. In particular, the sheaf $\mathcal{F}$ associated to an algebraic vector bundle is locally free: as the vector bundle is locally trivial, we have $F|_{U_i} \cong U_i \times \mathbb{C}^m$ for some open cover $\{U_i\}$ of X and so it follows that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus m}$ i.e. $\mathcal{F}$ is locally free. In particular, the category of vector bundles sits naturally inside the category of coherent algebraic sheaves as the subcategory of locally free sheaves. Because of the equivalence described above, we will interchangeably use the words vector bundle and locally free sheaf.

We briefly give some properties of sheaf cohomology that we will need below. By definition, the sheaf cohomology groups $H^i(X, \mathcal{F})$ are finite dimensional vector bundles that are obtained by taking the right derived functors of the global sections functor $\Gamma(X, -)$. We note that, in sharp contrast with the smooth case, the global sections of an algebraic vector bundle form a finite dimensional vector space. We can view elements of the 0th cohomology $H^0(X, \mathcal{F})$ as global sections of $\mathcal{F}$ or, equivalently, sheaf homomorphisms $\mathcal{O}_X \to \mathcal{F}$. As $\dim X = 1$, we have that $H^i(X, \mathcal{F}) = 0$, for $i > 1$. The 1st cohomology can be interpretted using Serre duality:

$$H^1(X, \mathcal{F})^* \cong H^0(X, \mathrm{Hom}(\mathcal{F}, \omega_X))$$

where ω_X is the canonical bundle (this is the cotangent bundle of X; it is a line bundle on X of degree $2g-2$). The Euler characteristic of a sheaf $\mathcal{F}$ over X is defined to be the alternating sum of the dimensions of the cohomology groups

$$\chi(\mathcal{F}) := \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F}).$$

The degree and rank are related by the Riemann-Roch formula:

$$\chi(\mathcal{F}) = \deg \mathcal{F} + \mathrm{rk}\, \mathcal{F} \chi(\mathcal{O}_X).$$

The Euler characteristic of the structure sheaf $\mathcal{O}_X$ is $\chi(\mathcal{O}_X) = 1 - g$ where g is the genus of X (since $H^0(X, \mathcal{O}_X) = 1$ and $H^1(X, \mathcal{O}_X) = g$).

Definition 7.1. A vector bundle $\mathcal{F}$ is generated by its global sections if the natural evaluation map

$$\mathrm{ev}_{\mathcal{F}} : H^0(X, \mathcal{F}) \otimes \mathcal{O} \to \mathcal{F}$$

is a surjection.

The notion of a vector bundle being generated by its global sections will be important in the construction we give below. In fact, we will prove that all semistable vector bundles of sufficiently large degree are generated by their global sections.

7.2. **Moduli spaces in algebraic geometry.** In algebraic geometry, a moduli problem is essentially a classification problem: we have a class of objects (for example, vector bundles) with an equivalence relation (such as isomorphism) and we want to describe the set of equivalence classes geometrically; i.e. in a way that encodes how the objects vary continuously. More precisely, we need to give a notion of families of objects parametrised by a base variety (or, more generally, a base scheme) S. Then we extend our equivalence relation to an equivalence relation $\sim_S$ for families over S. Furthermore, we want our notion of families to be functorial; that is, given a morphism $f : T \to S$ there should be a way to pullback families from S to T that is compatible with equivalence. Finally, we insist that a family parametrised by a point is just a single object (for example, a single vector bundle). This information is encoded by a moduli functor.

Definition 7.2. Let $\mathcal{S}ch$ denote the category of schemes and $\mathcal{S}et$ the category of sets. Then the moduli functor for a moduli problem is the contravariant functor $\mathcal{M} : \mathcal{S}ch \to \mathcal{S}et$ such that, for $S \in \mathcal{S}ch$,

$$\mathcal{M}(S) = \{\text{families parametrised by } S\}/\sim_S$$

The contravariance of $\mathcal{M}$ means that, for every morphism $f : S \to T$ in $\mathcal{S}ch$, we have a pullback

$$f^* : \mathcal{M}(T) \to \mathcal{M}(S).$$

Example 7.3 (Grassmannians)**.** Fix natural numbers $N > n$, then we can consider the moduli problem of classifying quotients $\mathbb{C}^N \to V$ where V has dimension n. We define two quotients to be equivalent if their kernels are the same. We can construct an associated moduli functor for this problem $\mathcal{G}r : \mathcal{S}ch \to \mathcal{S}et$ by defining a family over S to be a surjection from the trivial rank N-vector bundle $\mathcal{O}_S^{\oplus N}$ to a rank n vector bundle $\mathcal{Q}$ over S. The notion of equivalence of families over S is given by

$$(q : \mathcal{O}_S^{\oplus N} \to \mathcal{Q}) \sim (q' : \mathcal{O}_S^{\oplus N} \to \mathcal{Q}') \iff \ker q = \ker q'.$$

Then we define

$$\mathcal{G}r(S) = \{\text{vector bundle surjections } \mathcal{O}_S^{\oplus N} \to \mathcal{Q} \text{ over } S : \operatorname{rk} \mathcal{Q} = n\}/\sim$$

Given $f : T \to S$, we define the pullback by $f^*(\mathcal{O}_S^{\oplus N} \to \mathcal{Q}) := \mathcal{O}_T^{\oplus N} \to f^*\mathcal{Q}$.

Example 7.4 (algebraic vector bundles over a smooth complex projective algebraic curve X)**.** A family of vector bundles of rank n degree d parametrised by S is an algebraic vector bundle $\mathcal{E}$ over $S \times X$ of rank n such that, for all points $s \in S$, the vector bundle $\mathcal{E}_s := i_s^*\mathcal{E}$ on X has degree d where if $i_s : \{s\} \times X \to S \times X$ is the natural inclusion. We define an equivalence relation on vector bundles: two families $\mathcal{E}$ and $\mathcal{F}$ of vector bundles parametrised by S are equivalent if they are isomorphic as vector bundles over $X \times S$. For a morphism of schemes $f : S \to T$, the pullback of vector bundles along $f \times \mathrm{id}_X$ defines a map between the set of families over T and families over S. We let $\mathcal{M}_X(n, d)$ denote the associated moduli functor.

For any scheme $S \in \mathcal{S}ch$, we can consider its functor of points which is the covariant functor

$$h_S : \mathcal{S}ch \to \mathcal{S}et \qquad h_S(T) := \mathrm{Mor}(T, S).$$

We can now give the definition of a moduli space:

Definition 7.5. A scheme M is a fine moduli space for a moduli functor $\mathcal{M} : \mathcal{S}ch \to \mathcal{S}$ if there is a natural isomorphism of functors $\Phi : \mathcal{M} \to h_M$.

Therefore, for every $S \in \mathcal{S}$, we have a natural bijection

$$\Phi(S) : \mathcal{M}(S) \to h_M(S)$$

between equivalence classes of families over S and morphisms $f : S \to M$. In particular, if we take S to be a point, we see that there is a bijection between the set of all equivalence classes and points of M. If a fine moduli space exists, then the identity morphism $\mathrm{Id}_M : M \to M$ corresponds to a family $\mathcal{U} \in \mathcal{M}(M)$ parametrised by M called the universal family. The name universal family comes from the fact that, for any $f : S \to M$, the corresponding family parametrised by S is equivalent to the pullback of the universal family $f^*\mathcal{U}$.

Example 7.6. The grassmannian variety $\mathrm{Gr}(N, n)$ is a smooth projective variety of dimension $n(N - n)$ that is a fine moduli space for the Grassmannian moduli functor $\mathcal{G}r : \mathcal{S}ch \to \mathcal{S}et$ defined above. In particular, there is a universal quotient family

$$\mathcal{O}_{\mathrm{Gr}(N,n)}^{\oplus N} \to \mathcal{Q}$$

parametrised by $\mathrm{Gr}(N, n)$ and a morphism $f : S \to \mathrm{Gr}(N, n)$ is equivalent to a family parametrised by S i.e. a surjection $\mathcal{O}_S^{\oplus N} \to f^*\mathcal{Q}$ of vector bundles over S with $\operatorname{rk} f^*\mathcal{Q} = n$.

Many moduli problems have no fine moduli space and so we weaken the definition slightly.

Definition 7.7. A scheme M is a coarse moduli space for a moduli functor $\mathcal{M} : \mathcal{S}ch \to \mathcal{S}$ if there is a natural transformation of functors $\Phi : \mathcal{M} \to h_M$ such that

(1) $\Phi(\mathrm{pt})$ is bijective (i.e. the points of M parametrise equivalence classes);
(2) $\Phi : \mathcal{M} \to h_M$ is universal in the sense that any other scheme N and natural transformation $\Phi' : \mathcal{M} \to h_N$ factors uniquely via Φ (i.e. there exists $h_M \to h_N$ making a commutative triangle).

For coarse moduli spaces, there is no universal family and it is no longer the case that families correspond to morphisms to M. However, a family over S still defines a morphism $S \to M$.

Remark 7.8. A coarse moduli space M is a fine moduli space if and only if the natural transformation $\Phi : \mathcal{M} \to h_M$ is bijective; that is, if and only if:

(1) there is a family $\mathcal{U}$ parametrised by M such that, for all $m \in M$, the corresponding object $\mathcal{U}_m$ is equivalent to $\Phi(pt)^{-1}(m \hookrightarrow M)$;
(2) two families $\mathcal{F}$ and $\mathcal{F}'$ over S are equivalent if and only if they define the same map $S \to M$.

For details, see Newstead's notes [15], Proposition 1.8.

We weaken the notion of a universal family and search for a family with a local universal property.

Definition 7.9. A family $\mathcal{F}$ over a scheme S has the local universal property if for any other family $\mathcal{F}'$ parametrised by T and any point $t \in T$ there is a neighbourhood U of $t \in T$ and a morphism $f : U \to S$ such that the pullback $\mathcal{F}'|U$ of $\mathcal{F}'$ to U is equivalent to $f^*\mathcal{F}$.

Suppose we have a family $\mathcal{F}$ over S with a local universal property and a group G acting on S so that $\mathcal{F}_s \sim \mathcal{F}_{s'}$ if and only if s and t lie in the same orbit for this action. In this case, a natural candidate for a moduli space would be a quotient of the G-action on S and we can often use geometric invariant theory to construct such a quotient. We give the definition of a categorical quotient and a result that demonstrates the importance of categorical quotients in moduli problems before moving on to describe the main ideas of geometric invariant theory.

Definition 7.10. Let G be a group acting on a scheme S. A categorical quotient is a morphism $\pi : S \to Y$ of schemes that is G-invariant (i.e. constant on orbits) and is universal with this property; that is, any other G-invariant morphism $f : S \to Z$ factors uniquely through π. If in addition $\pi^{-1}(y)$ is a single orbit, for each $y \in Y$, then we call $\pi : S \to Y$ an orbit space.

Proposition 7.11 (Newstead [15], Proposition 2.13)**.** *Let $\mathcal{F}$ be a family over a scheme S with the local universal property for some moduli problem. If there is a group G acting on S so that $\mathcal{F}_s \sim \mathcal{F}_{s'}$ if and only if s and s' lie in the same G-orbit, then any coarse moduli space is a categorical quotient of S by G. Furthermore, a categorical quotient of S by G is a coarse moduli space if and only if its an orbit space.*

7.3. **GIT.** In this section, we outline the results of Mumford's geometric invariant theory (GIT) [10]. For our purposes, it suffices to work over the complex numbers so all varieties and schemes will be complex.

Let G be an algebraic group (i.e. an algebraic variety with a group structure such that multiplication and inversion are algebraic maps); for us, important examples are the groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$ and $\mathrm{PGL}(n)$ which are all smooth affine varieties. An algebraic action of G on an algebraic scheme V is an algebraic morphism $G \times V \to V$ that defines an action of G on V. Let $A(V)$ be the ring of regular functions on V (that is, morphisms $f : V \to \mathbb{C}$); then there is an induced action of G on $A(V)$ given by

$$(g \cdot f)(v) = f(g \cdot v).$$

We first describe the construction of the GIT quotient of an affine variety. This serves as an introduction to GIT and also explains the important role of reductive groups in this theory. Let $V \subset \mathbb{A}^n \cong \mathbb{C}^n$ be an affine variety defined as the zero locus of polynomials $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$. Then we can naturally identify the ring of regular functions on V with the finitely generated $\mathbb{C}$-algebra

$$A(V) \cong \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m).$$

A morphism of varieties $\phi : V \to Z$, induces a homomorphism of algebras

$$\phi^* : A(Z) \to A(V) \quad (f : Z \to \mathbb{C}) \mapsto (f \circ \phi : V \to \mathbb{C})$$

and if ϕ is G-invariant then, the image of this homomorphism lies in the invariant subring

$$A(V)^G := \{f \in A(V) : g \cdot f = f \text{ for all } g \in G\}.$$

In particular, if $A(V)^G$ is finitely generated, then a natural candidate for the categorical quotient of the G-action on V is the corresponding variety $\operatorname{Spec} A(V)^G$. The finite generation of $A(V)^G$ is important so that we can take finitely many generators $f_1, \dots, f_r$ of $A(V)^G$ and consider the surjection

$$\mathbb{C}[f_1, \dots, f_r] \to A(V)^G;$$

then $\operatorname{Spec} A(V)^G$ can be realised as a subvariety of $\mathbb{A}^r \cong \mathbb{C}^r$ defined by the zero locus of the polynomials in the ideal I which is the kernel of this surjection. In other words, the finite generation allows us to realise this as a subvariety of some affine space $\mathbb{A}^r$ for finite r. Therefore, in order to construct a categorical quotient, we consider the following question of Hilbert.

Hilbert's 14th problem Let V be an affine variety; then, for which groups G, is invariant subring $A(V)^G$ finitely generated as a $\mathbb{C}$-algebra?

Nagata proved that for reductive groups the invariant subring $A(V)^G$ is always finitely generated [12]. Fortunately, the groups we are interested in, namely $\mathrm{GL}(n)$, $\mathrm{SL}(n)$ and $\mathrm{PGL}(n)$, are all reductive.

Definition 7.12. Let G be a reductive algebraic group acting algebraically on an affine variety V, then the affine GIT quotient is the map $\pi : V \to V//G := \operatorname{Spec} A(V)^G$ corresponding to the inclusion of algebras $A(V)^G \subset A(V)$.

In particular, the GIT quotient of an affine variety is itself an affine variety.

Remark 7.13. The double slash notation $V//G$ used for the GIT quotient serves as a reminder that in general this is not an orbit space; that is, the quotient map $\pi : V \to V//G$ identifies some orbits. This identification happens when we have non-closed orbits: as the quotient map is continuous and constant on orbits it is also constant on orbit closures and so if an orbit is contained in the closure of another orbit, then these two orbits will be identified by the quotient. Moreover, this identification is often necessary to produce a seperated (i.e Hausdorff) quotient. We will see this phenomena in Example 7.15 below.

The important thing about the GIT quotient for us is the following theorem.

Theorem 7.14 (Mumford [10])**.** *Let G be a reductive algebraic group acting algebraically on an affine variety V, then the affine GIT quotient $\pi : V \to V//G$ is a G-invariant surjective map and is a categorical quotient of the G-action on V. Furthermore, if we consider the 'stable locus'*

$$V^s := \{v \in V : G \cdot v \text{ is closed of maximal dimension}\},$$

then there is an open subset $V^s/G \subset V//G$ such that $\pi^{-1}(V^s/G) = V^s$ and $\pi|_{V^s} : V^s \to V^s/G$ is an orbit space.

Example 7.15. The orbits for the action of $G = \mathbb{C}^*$ on $V = \mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$ are:

- conics $\{xy = \alpha\}$ for $\alpha \in \mathbb{C}^*$;
- the punctured x-axis;
- the punctured y-axis;
- the origin.

The origin and the conic orbits are closed orbits whereas the punctured axes both contain the origin in their orbit closures. Therefore, the last three orbits will all be identified by the GIT quotient. To determine the stable set, we note that the dimension of the orbit of the origin is strictly smaller than the dimension of $\mathbb{G}_m$ as its stabiliser has positive dimension. Hence $V^s = \{(x, y) \in \mathbb{A}^2 : xy \neq 0\}$. To construct the GIT quotient, we simply need to calculate the invariant subring of $A(V) = \mathbb{C}[x, y]$; there is a single invariant $A(V)^G = \mathbb{C}[xy]$. Therefore, $V//G = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[xy]$ and the GIT quotient $\pi : V \to V//G$ is given by $(x, y) \mapsto xy$. From the explicit description, we also see that the three orbits consisting of the punctured axes and the origin are all identified. If we restrict to the stable locus $\pi : V^s \to V^s/G = \mathbb{A}^1 - \{0\}$ then we get an orbit space.

More generally, Mumford defines a GIT quotient $S//G$ for a (linearised) reductive action on an algebraic scheme S. In this case, the categorical quotient is not a quotient of the whole of S but rather an open subset S^{ss} of 'semistable' points. The idea is that we want to try and cover S by suitable open affine subschemes U_i and then glue the affine GIT quotients $U_i//G$ to construct $S//G$; however, in general, it is not possible to cover all of S by suitable open affines U_i and so we can only cover an open subset (the semistable set) of S. The GIT quotient $\pi : S^{ss} \to S//G$ restricts to an orbit space on an open subset of stable points S^s and we have open inclusions $S^s \subset S^{ss} \subset S$.

In the affine case described above, all points are semistable i.e $V = V^{ss}$ and the GIT quotient $V//G$ is also affine. Another important feature of GIT is that the GIT quotient of a projective scheme is itself projective.

7.4. **Boundedness of semistable bundles.** Ideally, to construct a moduli space of vector bundles on X using GIT, we would like to find a variety (or scheme) P that parametrises a family $\mathcal{F}$ of vector bundles on X of fixed rank n and degree d such that any vector bundle of the given invariants is isomorphic to $\mathcal{F}_p$ for some $p \in P$. Unfortunately, the family of all vector bundles on X of rank $n > 1$ and degree d is too large to be parametrised by such a scheme P (we say the family of all vector bundles on X of fixed rank and degree is unbounded). In this section, we will see that the family of semistable vector bundles on X of fixed rank and degree is bounded (i.e. there exists such a parameter scheme P when we restrict to semistable bundles of fixed invariants).

First, we note that we can assume the degree of our vector bundle is sufficiently large: for, if we take a line bundle $\mathcal{L}$ of degree e, then tensoring with $\mathcal{L}$ preserves (semi)stability and so induces an isomorphism between moduli of (semi)stable vector bundles with rank and degree (n, d) and those with rank and degree $(n, d + ne)$

$$- \otimes \mathcal{L} : \mathcal{M}(n, d) \cong \mathcal{M}(n, d + ne).$$

Hence, we can assume that $d > n(2g - 1)$ where g is the genus of X. We make this assumption in order to prove the following lemma that is essential for the proof of boundedness.

Lemma 7.16. *Let $\mathcal{F}$ be a semistable locally free sheaf (i.e. vector bundle) of rank n and degree $d > n(2g - 1)$ on X. Then*

1. *$\mathcal{F}$ is generated by its global sections;*
2. *$H^1(X, \mathcal{F}) = 0$.*

Proof. For the second part we use Serre duality: if $H^1(X, \mathcal{F}) \neq 0$, then dually there would be a non-zero homomorphism $f : \mathcal{F} \to \omega_X$. We let $\mathcal{E} \subset \mathcal{F}$ be the subbundle generically generated by the kernel of f which is a subbundle of rank $n - 1$ and degree

$$\deg \mathcal{E} \geq \deg \ker f \geq \deg \mathcal{F} - \deg \omega_X = d - (2g - 2).$$

In this case, by semistability of $\mathcal{F}$, we have

$$\frac{d - (2g - 2)}{n - 1} \leq \mu(\mathcal{E}) \leq \mu(\mathcal{F}) = \frac{d}{n}$$

and thus $d \leq n(2g - 2)$.

The first part requires some more involved sheaf theory. To show that $\mathcal{F}$ is generated by its sections, we will show that for each $x \in X$, the map $H^0(X, \mathcal{F}) \to F_x$ to the fibre F_x over x of the corresponding vector bundle F is surjective. For this, we use the fact that there is a short exact sequence of sheaves

$$0 \to \mathcal{I}_x \otimes \mathcal{F} \to \mathcal{F} \to F_x \to 0$$

where F_x is viewed as a torsion sheaf supported on x and $\mathcal{I}_x = \mathcal{O}_X(-x)$ is the sheaf of ideals defining the point x (which is a line bundle on X of degree -1). Then we can consider the associated long exact sequence in cohomology

$$0 \to H^0(X, \mathcal{I}_x \otimes \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(F_x) \to H^1(X, \mathcal{I}_x \otimes \mathcal{F}) \to \cdots .$$

To show that $H^0(X,\mathcal{F}) \to H^0(F_x) = F_x$ is surjective, is equivalent to showing $H^1(X, \mathcal{I}_x \otimes \mathcal{F}) = 0$. Now we use the argument above: we know that $\mathcal{I}_x \otimes \mathcal{F}$ is also semistable (tensoring with a line bundle preserves semistability) and

$$\deg \mathcal{I}_x \otimes \mathcal{F} = d - r > r(2g-2)$$

and so it follows from above that $H^1(X, \mathcal{I}_x \otimes \mathcal{F}) = 0$. □

As mentioned above, these two properties are important for showing boundedness. In fact, we will see that a strictly larger family of vector bundles of fixed rank and degree are bounded; namely those that are generated by their global sections and have vanishing 1st cohomology. Given a vector bundle $\mathcal{F}$ of rank n and degree d that is generated by its global sections, we can consider the evaluation map

$$\mathrm{ev}_{\mathcal{F}} : H^0(X,\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F}$$

which is, by assumption, surjective. If also $H^1(X,\mathcal{F}) = 0$, then

$$\chi(\mathcal{F}) = d + n(1-g) = \dim H^0(X,\mathcal{F}) - \dim H^1(X,\mathcal{F}) = \dim H^0(X,\mathcal{F});$$

that is, the dimension of the 0th cohomology is fixed and equal to $N := d + n(1-g)$. We can choose an isomorphism $H^0(X,\mathcal{F}) \cong \mathbb{C}^N$ and combine this with the evaluation map for $\mathcal{F}$, to produce a surjection

$$q_{\mathcal{F}} : \mathcal{O}_X^{\oplus N} = \mathbb{C}^N \otimes \mathcal{O}_X \to \mathcal{F}.$$

Equivalently, we can think of this surjection as corresponding to a morphism $X \to \mathrm{Gr}(N,n)$ by Example 7.6. As $\mathcal{F}$ has degree d, this is a degree d morphism. In this way we see that every vector bundle of fixed rank and degree that is generated by its global sections and has zero 1st cohomology corresponds to a degree d map $X \to \mathrm{Gr}(N,n)$ and, in particular, corresponds to a point in the set

$$\mathrm{Map}_d(X, \mathrm{Gr}(N,n))$$

of degree d maps from X to $\mathrm{Gr}(N,n)$. This set $\mathrm{Map}_d(X,\mathrm{Gr}(N,n))$ has the structure of a scheme: it can be realised as a subscheme of the Hilbert scheme parametrising subschemes of the projective variety $X \times \mathrm{Gr}(N,n)$ (actually, here it is important that both X and $\mathrm{Gr}(N,n)$ are projective so that their automorphism groups are not too large). We recall that the Grassmannian is a fine moduli space with universal quotient family

$$\mathcal{O}_{\mathrm{Gr}(N,n)}^{\oplus N} \to \mathcal{Q}$$

and so every degree d map $f : X \to \mathrm{Gr}(N,n)$ corresponds to a quotient $\mathcal{O}_X^{\oplus N} \to f^*\mathcal{Q}$ where $f^*\mathcal{Q}$ has rank n and degree d.

In fact, it suffices to consider the open subset $T \subset \mathrm{Map}_d(X,\mathrm{Gr}(N,n))$ consisting of degree d maps $f : X \to \mathrm{Gr}(N,n)$ such that

(1) The natural map $H^0(X, \mathcal{O}_X^{\oplus N}) \to H^0(X, f^*\mathcal{Q})$ is an isomorphism;
(2) $H^1(X, f^*\mathcal{Q}) = 0$.

We can think of T as parametrising a family of vector bundles of rank n and degree d that are generated by their global sections and have vanishing 1st cohomology. In fact, Seshadri proves that this family has the local universal property. Since we are only interested in semistable sheaves, we can consider the open sets $S^{(\mathrm{s})\mathrm{s}} \subset T$ of maps $f : X \to \mathrm{Gr}(N,n)$ such that $f^*\mathcal{Q}$ is a (semi)stable vector bundle on X. Thus S^{s} (resp. S^{ss}) parametrises a family of stable (resp. semistable) vector bundles of rank n and degree d on X. In particular, we have proved the first part of the following propostion; for the second part, see [23] Proposition 6.2.

Proposition 7.17. *Let $n \in \mathbb{N}$ and $d > n(2g-1)$ where g is the genus of X. Then the set of (semi)stable sheaves of rank n and degree d on X is bounded. In particular, the scheme $S^{(s)s}$ parametrises a family of (semi)stable vector bundles of rank n and degree d on X with the local universal property.*

7.5. **Construction of the moduli space.** Let $S^{(s)s} \subset T$ be the subschemes considered above parametrising families of (semi)stable vector bundles of rank n and degree d on X with the local universal property. We can ask whether there is natural group action on these schemes such that two points lie in the same orbit if and only if they represent isomorphic vector bundles. We recall that to associate to a semistable vector bundle $\mathcal{F}$ a point in the parameter space S^{ss}, we had to choose an isomorphism $H^0(X,\mathcal{F}) \cong \mathbb{C}^N$ and as the group $\mathrm{GL}(N)$ relates any two such choices, we should naturally consider the induced action of $\mathrm{GL}(N)$ on $\mathrm{Gr}(N,n)$ and $\mathrm{Map}_d(X,\mathrm{Gr}(N,n))$. Indeed, the orbits for this action correspond to isomorphism classes of vector bundles because the bundles are generated by their global sections. Since the subgroup $\mathbb{C}^* I_N \subset \mathrm{GL}(N)$ acts trivially, we instead consider the induced action of $\mathrm{PGL}(N)$.

We are now in the usual set-up for a GIT construction of a moduli space. More precisely, we have a scheme T that parametrises a family of degree d and rank n vector bundles on X with the local universal property and a reductive group $\mathrm{PGL}(N)$ acting on T such that the orbits correspond to isomorphism classes of vector bundles. The idea is to apply Proposition 7.11. Using the theory of GIT, we can construct a categorical quotient of the GIT semistable set T^{ss} and an orbit space of the GIT stable set T^{s}. We want to relate the notions of GIT (semi)stability with the notions of vector bundle (semi)stability. As mentioned above, Mumford's notion of vector bundle (semi)stability came from his work on GIT and so naturally we have the following result.

Theorem 7.18 (Seshadri [23])**.** *We have that $T^s = S^s$ and $T^{ss} = S^{ss}$.*

In other words, the notions of GIT (semi)stability and vector bundle (semi)stability coincide. The natural candidate for a moduli space is then the GIT quotient for $\mathrm{PGL}(N)$-action on T. By Mumford's theory of GIT, the GIT quotient

$$T//\mathrm{PGL}(N)$$

is a categorical quotient of T^{ss} and, moreover, it contains an open subset $T^s/\mathrm{PGL}(N)$ that is an orbit space for the action on T^s. Seshadri shows that this GIT quotient is a projective variety (and so can be thought of as a completion of $T^s/\mathrm{PGL}(N)$) by realising this GIT quotient as a projective GIT quotient. Furthermore, by studying the deformation theory of semistable vector bundles, he shows that this GIT quotient is smooth of dimension $n^2(g-1)+1$ (here it is important that X is smooth of dimension 1).

Theorem 7.19 (Newstead [15], Seshadri [23])**.** *There exists a coarse moduli space $M^s(n,d)$ of isomorphism classes of stable vector bundles of rank r and degree d on X that has a natural completion to a smooth complex projective variety $M^{ss}(n,d)$ of dimension $n^2(g-1)+1$.*

Proof. We can assume without loss of generality that d is sufficiently large so that the set of semistable sheaves with rank n and degree d are bounded; then we let $T^s \subset T^{ss} \subset T$ be the spaces constructed above. The idea is to construct the moduli space using GIT. We let $M^s(n,d) := T^s/\mathrm{PGL}(N)$ and $M^{ss}(n,d) := T//\mathrm{PGL}(N)$. By Mumford's results on GIT, the quotient of the GIT stable locus $T^s/\mathrm{PGL}(N)$ is an orbit space for the $PGL(N)$-action on T^s. Since also T^s parametrises a family of stable vector bundles on X of rank n and degree d with the local universal property, we apply Proposition 7.11 to complete the proof. □

If the degree and rank are coprime, then it's easy to see from the definition of (semi)stability that the notions of semistability and stability coincide. In particular, the moduli space of stable vector bundles of rank r and degree d on X is a smooth projective variety.

Remark 7.20. The GIT quotient $M^{ss}(n,d) := T//\mathrm{PGL}(N)$ can be seen as a moduli space of polystable vector bundle as every semistable vector bundle contains a unique polystable vector bundle in its orbit closure.

Finally, we ask whether this coarse moduli space is ever a fine moduli space. To answer this question satisfactorily, it is necessary to modify our notion of equivalence of families. We originally defined two families of vector bundles parametrised by S to be isomorphic if they are isomorphic as vector bundles over $X \times S$. However, two families $\mathcal{E}$ and $\mathcal{F}$ parametrised by

S determine the same morphism to $M^s(n,d)$ if $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* \mathcal{L}$ for a line bundle $\mathcal{L}$ on S where $\pi_S : S \times X \to S$ is the projection map and, in fact, this is an if and only if statement (cf. [15] Lemma 5.10). Therefore, we define a new notion of equivalence of families over S: we say that $\mathcal{E} \sim_S \mathcal{F}$ if there is a line bundle $\mathcal{L}$ on S such that $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* \mathcal{L}$.

With this new equivalence relation for families in place, it is a result of Mumford and Newstead, for $n = 2$, [11] and Tjurin [25] in general that the moduli space of stable vector bundles is a fine moduli space for coprime rank and degree.

Theorem 7.21. *If $(n,d) = 1$, then $M^s(n,d) = M^{ss}(n,d)$ is a fine moduli space.*

The idea of the proof is to construct a universal family over this moduli space by descending the family of vector bundles parametrised by T^s to the quotient $T^s/\mathrm{PGL}(N)$. In order to do this we want to lift the $\mathrm{PGL}(N)$-action to this family parametrised by T^s. In general, this is not possible, but if we take an equivalent family parametrised by T^s (i.e. by tensoring with the pullback of a line bundle over T^s), then in the coprime case the $\mathrm{PGL}(N)$-action lifts to this equivalent family. In fact, for the construction of the desired line bundle on T^s, one needs (n,d) to be coprime. For more details, we recommend the exposition given by Newstead [15], Theorem 5.12.

Remark 7.22. If $(n,d) \neq 1$, then Ramanan observes that a fine moduli space for stable sheaves does not exist [17].

8. Non-abelian Hodge correspondence for the unitary group

We summarise the above results by stating the non-abelian Hodge correspondence over a compact Riemann surface X for the compact group $G = U(n)$. This is a special case of a much more general correspondence due to work of Corlette, Donaldson, Hitchin and Simpson; the special case of $\mathrm{U}(n)$ follows from work of Atiyah and Bott [1], Donaldson [3] and Narasimhan and Seshadri [13]. We start by recalling that the (abelian) Hodge theorem relates three cohomology groups associated to X:

(1) the de Rham cohomology groups $H^k_{dR}(X,\mathbb{C})$;
(2) the singilar (Betti) cohomology groups $H^k_B(X,\mathbb{C})$;
(3) the Dolbeault cohomology groups $H^k_{Dol}(X,\mathbb{C}) := \oplus H^p(X,\Omega^q)$ where the sum is taken over p,q such that $p+q=k$.

As X is a smooth manifold, the de Rham and Betti cohomology groups agree by the de Rham Theorem. Then, as X is also a Kähler manifold, the Dolbeault and de Rham cohomology groups agree by the Hodge Theorem. Hence, for a Kähler manifold all three cohomology groups coincide. We can think of the non-abelian Hodge theorem as the analogous statment for when we consider coefficients in $\mathrm{U}(n)$. The points in the 1st singular cohomology group of X with coefficients in $\mathrm{U}(n)$ can be thought of as conjugacy classes of representations of the fundamental group to X (this is the Betti moduli space). The analogous de Rham, Betti and Dolbeault moduli spaces are defined as follows.

The de Rham moduli space

$$M_{dR} := \mu^{-1}\left(\frac{id}{n} I\right)/\mathcal{G}$$

of gauge equivalence classes of projectively flat unitary connections on a Hermitian vector bundle over X of rank n and degree d. Here $\mu : \mathcal{A} \to \mathrm{Lie}\,\mathcal{G}^*$ is the moment map for the gauge group action given by taking a connection to (minus) its curvature 2-form. This moduli space is constructed as an infinite dimensional symplectic reduction.

The Betti moduli space (or twisted character variety)

$$M_B := \frac{\mathrm{Hom}^d(\hat{\pi}_1(X)_{\mathbb{R}}, \mathrm{U}(n))}{\mathrm{U}(n)} := \frac{\mu_g^{-1}\left(\exp\left(2\pi i d/n\right) I_n\right)}{\mathrm{U}(n)}$$

of d-twisted representations of the fundamental group of X into $\mathrm{U}(n)$. Here $\mu_g : \mathrm{U}(n)^{2g} \to \mathrm{U}(n)$ is the map given by $\mu_g(A_1, \dots, B_g) = \prod_{i=1}^g [A_i, B_i]$. In the coprime case, we can take the usual smooth quotient; however, in the non-comprime case, we must identify some reduced representations in order to get a Hausdorff quotient.

The Dolbeault moduli space

$$M_{Dol} := M^s(n, d)$$

of (poly)stable algebraic vector bundles on X of rank n and degree d. The Dolbeault moduli space is constructed as a projective GIT quotient and is a smooth complex projective variety of dimension $n^2(g-1)+1$. In the coprime case, the notions of semistability and stability coincide and so this is the moduli space of stable algebraic vector bundles.

The non-abelian Hodge correspondence is the culmination of the results we have seen above. More precisely, the Narasimhan–Seshadri–Donaldson correspondence describes a correspondence between (projectively) flat unitary connections, (twisted) unitary representations of the fundamental group and polystable algebraic vector bundles; that is, we have a correspondence between the objects parambetrised by these three moduli spaces.

Theorem 8.1 (Non–Abelian Hodge correspondence)**.** *Let* $(n, d) = 1$*; then we have identifications*

$$M_{dR} \cong M_B \cong M_{Dol}$$

where the first is an isomorphism of real analytic spaces given by the Riemann–Hilbert corresponce and the second is a homeomorphism of topological spaces given by the Narasimhan–Seshadri correspondence.

Example 8.2. We describe this correspondence carefully for the case of $n = 1$ and $d = 0$. For the de Rham moduli space, we want to consider gauge equivalence classes of unitary connections on the trivial line bundle. The space of connections on the trivial line bundle is an affine space modeled on $\Omega^1(X, \mathrm{End}(X \times \mathbb{C}))$ and the space of unitary connections is the space of 1-forms $\Omega^1(X)$. A connection A is flat if and only if $dA = 0$; that is, the connection 1-form is closed. As $\mathrm{U}(1)$ is commutative, the adjoint action is trivial; in particular, this means that

$$\mathcal{G} := \mathrm{Hom}^{\mathrm{U}(1)}(X \times \mathrm{U}(1), \mathrm{U}(1)) = \mathrm{Hom}(X, \mathrm{U}(1))$$

and similarly, $\mathrm{Lie}\,\mathcal{G} := \mathrm{Hom}^{\mathrm{U}(1)}(X \times \mathrm{U}(1), \mathbb{R}) = \mathrm{Hom}(X, \mathbb{R}) = \Omega^0(X)$. Infinitesimally, $\Psi \in \mathrm{Lie}\,\mathcal{G} = \Omega^0(X)$ acts by addition by the 1-form $d\Psi$. Hence, the space of flat connections modulo the infinitesimal action is

$$\frac{\mathcal{A}_{\mathrm{flat}}}{\exp \mathrm{Lie}\,\mathcal{G}} = \frac{\{A \in \Omega^1(X) : dA = 0\}}{\{d\Psi : \Psi \in \Omega^0(X)\}} = H^1_{dR}(X, \mathbb{R}) = \mathbb{R}^{2g}.$$

The obstruction to lifting an element $\Phi \in \mathcal{G}\mathrm{Hom}(X, \mathrm{U}(1))$ to a map $\Psi : X \to \mathbb{R}$ is $H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$. Therefore,

$$M_{dR} := \frac{\mathcal{A}_{\mathrm{flat}}}{\mathcal{G}} = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})} = \frac{\mathbb{R}^{2g}}{\mathbb{Z}^{2g}} = (S^1)^{2g}.$$

For the Betti moduli space, we consider conjugacy classes of representations of the fundamental group into $\mathrm{U}(1)$. As $\mathrm{U}(1)$ is abelian, the conjugation $\mathrm{U}(1)$ action is trivial and $\mu_g : \mathrm{U}(1)^{2g} \to \mathrm{U}(1)$ has image equal to the identity (i.e. all representations of the free group on $2g$ elements define a representation of $\pi_1(X)$). Therefore, the Betti moduli space is a real torus:

$$M_B := \frac{\mathrm{Hom}(\pi_1(X), \mathrm{U}(1))}{\mathrm{U}(1)} = \mathrm{U}(1)^{2g}.$$

The Riemann–Hilbert correspondences gives a real analytic isomorphism

$$M_B := \mathrm{U}(1)^{2g} \cong (S^1)^{2g} =: M_{dR}.$$

As every line bundle is stable, the Dolbeault moduli space is the Jacobian of X (the Jacobian is a fine moduli space for degree 0 line bundles on X). By definition, the Jacobian is a complex torus constructed as a quotient of a complex vector space by a lattice

$$M_{Dol} = \mathrm{Jac}^0(X) := \frac{\Gamma(X,\Omega^1)^\vee}{H_1(X,\mathbb{Z})} = \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}$$

where here Ω^1 is the sheaf of holomorphic 1-forms and the inclusion $H_1(X,\mathbb{Z}) \hookrightarrow \Gamma(X,\Omega^1)^\vee$ is given by sending a loop $\gamma \in H^1(X,\mathbb{Z})$ to $\omega \mapsto \int_\gamma \omega$. To justify why the Jacobian is a moduli space for degree 0 line bundles on X, we recall that line bundles on the compact Riemann surface X correspond to divisors $D := \sum n_i P_i$ (i.e. finite integral sums of points $P_i \in X$). More precisely, every divisor D on X defines a line bundle $\mathcal{L}(D)$ on X whose sections are meromorphic functions on X with poles controlled by D. A special class of divisors are the principal divisors $\mathrm{div} f$ associated to a meromorphic function f on X; by definition, $\mathrm{div} f := \sum_p \mathrm{ord}_P(f)P$ where $\mathrm{ord}_P(f) \in \mathbb{Z}$ is the order of f at P. Furthermore, two divisors D, D' define isomorphic line bundles if and only if their difference is a principal divisor (we write $D \sim D'$). The degree of the line bundle $\mathcal{L}(D)$ is the degree of the divisor $\deg D := \sum n_i$. If $\mathrm{Div}_0(X)$ denotes the degree zero divisors on X, then the set $\mathrm{Div}_0(X)/\sim$ is in bijection with the set of isomorphism classes of degree zero line bundles. To realise this as a complex tori, we use the Abel-Jacobi map:

$$\mathrm{Div}_0(X) \to \mathrm{Jac}(X) \qquad \sum n_i P_i \mapsto \left(\omega \mapsto \sum n_i \int_{\gamma_i} \omega \mod H_1(X,\mathbb{Z})\right)$$

where γ_i is a path from a fixed base point $x \in X$ to $P_i \in X$ (we need to choose this path and so this map is only defined modulo the lattice $H_1(X,\mathbb{Z})$). Abel proves this map is a surjection and the kernel is the set of principal divisors.

The Narasimhan–Seshadri correspondence gives a homeomorphism

$$M_B := \mathrm{U}(1)^{2g} \cong M_{Dol} := \mathbb{C}^g/\mathbb{Z}^{2g}$$

between the smooth real Betti moduli space and the smooth complex Dolbeault moduli space.

Remark 8.3. We conclude these notes with a few final comments.

(1) The natural bijection between points in the de Rham and Dolbeault moduli spaces is not continuous (cf. [24]).
(2) The homeomorphism defined by the Narasimhan–Seshadri correspondence can be thought of as an (infinite dimensional) example of the Kempf-Ness theorem [5] that gives a homeomorphism between a GIT quotient $V//\mathrm{G}_\mathbb{C}$ of a complex projective variety V by a complex reductive group $G_\mathbb{C}$ and the symplectic reduction for the compact group G acting on V. The idea is to show that the preimage of zero under the moment map $\mu^{-1}(0)$ is contained in the GIT semistable set V^{ss} and so this induces a map from the symplectic reduction to the GIT quotient.
(3) The full version of the non-abelian Hodge correspondence is due to work of Corlette, Donaldson, Hitchin and Simpson and studies moduli spaces associated to a (not necessarily compact) Lie group G. In this more general framework, moduli spaces of G-Higgs bundles appear as the Dolbeault moduli spaces.

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