GROUP ACTIONS ON QUIVER VARIETIES AND APPLICATIONS

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Abstract. We study two types of actions on moduli spaces of quiver representations over a field k and we decompose their fixed loci using group cohomology. First, for a perfect field k, we study the action of the absolute Galois group of k on the $\overline{k}$-valued points of this quiver moduli space; the fixed locus is the set of k-rational points and we obtain a decomposition of this fixed locus indexed by elements in the Brauer group of k. Second, we study algebraic actions of finite groups of quiver automorphisms on these moduli spaces; the fixed locus is decomposed using group cohomology and we give a modular interpretation of each component. As an application, we construct branes in hyperkähler quiver varieties, as fixed loci of such actions.

Contents

1. Introduction

For a quiver Q and a field k, we consider moduli spaces of semistable k-representations of Q of fixed dimension $d \in \mathbb{N}^V$, which were first constructed for an algebraically closed field k using geometric invariant theory (GIT) by King [19]. For an arbitrary field k, one can use Seshadri's extension of Mumford's GIT to construct these moduli spaces. More precisely, these moduli spaces are constructed as a GIT quotient of a reductive group $\mathbf{G}_{Q,d}$ acting on an affine space $\mathrm{Rep}_{Q,d}$ with respect to a character χ_θ determined by a stability parameter $\theta \in \mathbb{Z}^V$. The stability parameter also determines a slope-type notion of θ-(semi)stability for k-representations of Q, which involves testing an inequality for all proper non-zero

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subrepresentations. When working over a non-algebraically closed field, the notion of θ-stability is no longer preserved by base field extension, and so one must instead consider θ-geometrically stable representations (that is, representations which are θ-stable after any base field extension), which correspond to the GIT stable points in $\mathrm{Rep}_{Q,d}$ with respect to χ_θ. We let $\mathcal{M}_{Q,d}^{\theta-ss}$ (resp. $\mathcal{M}_{Q,d}^{\theta-gs}$) denote the moduli space of θ-semistable (resp. θ-geometrically stable) k-representations of Q of dimension d; these are both quasi-projective varieties over k.

We study two types of actions on these moduli spaces. First, for a perfect field k, we study the action of the absolute Galois group $\mathrm{Gal}_k = \mathrm{Gal}(\overline{k}/k)$ on $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$, whose fixed locus is the set $\mathcal{M}_{Q,d}^{\theta-ss}(k)$ of k-rational points. Second, we define a finite group $\mathrm{Aut}(Q)$ of covariant and contravariant automorphisms of Q in §4 and consider algebraic actions of subgroups $\Sigma \subset \mathrm{Aut}(Q)$ on $\mathcal{M}_{Q,d}^{\theta-ss}$, when the stability parameter θ and dimension vector d are compatible with the Σ-action in the sense of §4. We restrict the action of Gal_k (resp. Σ) to $\mathcal{M}_{Q,d}^{\theta-gs} \subset \mathcal{M}_{Q,d}^{\theta-ss}$, so we can use the fact that the stabiliser of every GIT stable point in $\mathrm{Rep}_{Q,d}$ is a diagonal copy of $\mathbb{G}_m$, denoted Δ, in $\mathbf{G}_{Q,d}$ (*cf.* Corollary 2.13) to decompose the fixed locus of Gal_k (resp. Σ) acting on $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ (resp. $\mathcal{M}_{Q,d}^{\theta-gs}$) in terms of the group cohomology of Gal_k (resp. Σ) with values in Δ or the (non-Abelian) group $\mathbf{G}_{Q,d}$.

Our methods and results have a similar flavour for both actions; however, Galois actions turn out to be slightly simpler than quiver automorphism groups, due to Hilbert's 90th Theorem, as we will explain below. Let us outline the main steps involved in the decomposition of these fixed loci.

As the action of Gal_k (resp. Σ) on $\mathcal{M}_{Q,d}^{\theta-ss}$ can be induced by compatible actions of this group on $\mathrm{Rep}_{Q,d}$ and $\mathbf{G}_{Q,d}$, the first step to study the fixed locus is to construct a map of sets

$$f_{\mathrm{Gal}_k} : \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k}/\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k} \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k} = \mathcal{M}_{Q,d}^{\theta-gs}(k)$$

in (3.5) and a morphism $f_\Sigma : \mathrm{Rep}_{Q,d}^{\Sigma} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^\Sigma$ of k-varieties in Propositions 4.8 and 4.37. In the Galois setting, the non-empty fibres of f_{Gal_k} are in bijection with the kernel of $H^1(\mathrm{Gal}_k, \Delta(\overline{k})) \longrightarrow H^1(\mathrm{Gal}_k, G(\overline{k}))$ by Proposition 3.3, and so by Hilbert's 90th Theorem, we deduce that f_{Gal_k} is injective (*cf.* Corollary 3.4). For a group Σ of quiver automorphisms, we also obtain an analogous description of the non-empty geometric fibres of the restriction f_Σ^{rs} of f_Σ to the preimage of $\mathcal{M}_{Q,d}^{\theta-gs}$ (*cf.* Proposition 4.10); however f_Σ need not be injective on geometric points (*cf.* Example 4.24).

The second step is to construct a so-called type map from the fixed set of the moduli space of θ-geometrically stable representations to the second group cohomology of Gal_k (resp. Σ) with values in Δ. More precisely, for the Gal_k-action, we have a type map $\mathcal{T} : \mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k} \longrightarrow H^2(\mathrm{Gal}_k, \Delta(\overline{k})) \cong \mathrm{Br}(k)$ taking values in the Brauer group $\mathrm{Br}(k)$ of k by Proposition 3.6 and, for $\Sigma \subset \mathrm{Aut}(Q)$ and each algebraically closed field Ω/k, we have a type map $\mathcal{T}_\Omega : \mathcal{M}_{Q,d}^{\theta-gs}(\Omega)^\Sigma \longrightarrow H^2(\Sigma, \Delta(\Omega))$, by Proposition 4.12 (see also §4.5). We prove that the image of f_{Gal_k} is $\mathcal{T}^{-1}([1])$ in Theorem 3.8; however, the image of $f_\Sigma^{rs}(\Omega)$ is, in general, strictly contained in the preimage of the trivial element under the type map (*cf.* Example 4.23).

The third step is to introduce the notion of a modifying family u (*cf.* Definitions 3.10 and 4.14) in order to modify the action of Gal_k (resp. Σ) on $\mathrm{Rep}_{Q,d}(\overline{k})$ and $\mathbf{G}_{Q,d}(\overline{k})$ (resp. $\mathrm{Rep}_{Q,d}$ and $\mathbf{G}_{Q,d}$) such that the induced action of Gal_k (resp. Σ)

on the moduli space $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ (resp. $\mathcal{M}_{Q,d}^{\theta-ss}$) coincides with the original action. Then we define a map of sets

$$f_{\mathrm{Gal}_k,u} : \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k,u}/\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u} \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k} = \mathcal{M}_{Q,d}^{\theta-gs}(k),$$

where $\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k,u}$ and $\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$ denote the fixed loci for the modified Gal_k-action given by u. Similarly, for a group Σ of covariant automorphisms of Q, we construct a morphism $f_{\Sigma,u} : \mathrm{Rep}_{Q,d}^{\Sigma,u}/\!/_{\chi_\theta}\mathbf{G}_{Q,d}^{\Sigma,u} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^{\Sigma}$ of k-varieties associated to the modified Σ-action given by u. Moreover, we show that the image of $f_{\mathrm{Gal}_k,u}$ (resp. $f_{\Sigma,u}^{rs}$) is equal to (resp. contained in) the preimage under the type map of the cohomology class of a Δ-valued 2-cocycle c_u naturally defined by u (*cf.* Theorems 3.12 and 4.17).

The fourth step is to describe the modifying families using the group cohomology of Gal_k (resp. Σ) in order to obtain a decomposition of the fixed locus for the Gal_k-action (resp. Σ-action) on the moduli space of θ-geometrically stable representations. Let us first state our decomposition theorem for Galois actions.

Theorem 1.1. *Let k be perfect and let $\mathcal{T} : \mathcal{M}_{Q,d}^{\theta-gs}(k) \longrightarrow H^2(\mathrm{Gal}_k; \overline{k}^{\times}) \cong \mathrm{Br}(k)$ be the type map introduced in Proposition 3.6. Then there is a decomposition*

$$\mathcal{M}_{Q,d}^{\theta-gs}(k) \simeq \bigsqcup_{[c_u]\in \mathrm{Im}\,\mathcal{T}} \mathrm{Rep}_{Q,d}^{\chi_\theta-s}(\overline{k})^{\mathrm{Gal}_k,u}/\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$$

where $\mathrm{Rep}_{Q,d}^{\chi_\theta-s}(\overline{k})^{\mathrm{Gal}_k,u}/\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$ is the set of isomorphism classes of θ-geometrically stable d-dimensional representations of Q that are k-rational with respect to the twisted Gal_k-action Φ^u on $\mathrm{Rep}_{Q,d}(\overline{k})$ defined in Proposition 3.11.

For example, if $k = \mathbb{R} \hookrightarrow \overline{k} = \mathbb{C}$, then as $\mathrm{Br}(\mathbb{R}) = \{\pm 1\}$, there are two types of rational points in $\mathcal{M}_{Q,d}^{\theta-gs}(\mathbb{R})$, namely $\mathbb{R}$-representations of Q, corresponding to $1 \in \mathrm{Br}(\mathbb{R})$, and 'quaternionic' representations of Q (*cf.* Example 3.13), corresponding to $-1 \in \mathrm{Br}(\mathbb{R})$.

Our decomposition result for algebraic actions of $\Sigma \subset \mathrm{Aut}(Q)$ is more complicated, as there is no analogue of Hilbert's 90th Theorem, so the images of the morphisms $f_{\Sigma,u}$ may not be equal to the whole of $\mathcal{T}^{-1}([c_u])$ and these morphisms may not be injective (*cf.* Examples 4.23 and 4.24 respectively). However, the fibres of these new morphisms can be described, again by methods of group cohomology.

Theorem 1.2. *Let k be an algebraically closed field and let $\Sigma \subset \mathrm{Aut}(Q)$ be a subgroup of covariant automorphisms of Q for which θ and d are compatible. Then there is a decomposition of the Σ-fixed locus*

$$(\mathcal{M}_{Q,d}^{\theta-s})^{\Sigma} = \bigsqcup_{\substack{[c_u]\in \mathrm{Im}\,\mathcal{T} \\ \overline{[b]}\in H^1_u(\Sigma,\mathbf{G}_{Q,d}(k))/H^1(\Sigma,\Delta(k))}} \mathrm{Im}\, f_{\Sigma,u^b}^{rs}$$

where u^b is the modifying family determined by $[b]$ and u via Lemma 4.19, and f_{Σ,u^b}^{rs} is the morphism defined as in Theorem 4.17.

Moreover, non-empty fibres of f_{Σ,u^b}^{rs} are in bijection with the pointed set

$$\ker\left(H^1(\Sigma;\Delta(\Omega)) \longrightarrow H^1_{u^b}(\Sigma,\mathbf{G}_{Q,d}(\Omega))\right).$$

If k is not algebraically closed, then one can get a decomposition of the Ω-valued points of $\mathcal{M}_{Q,d}^{\theta-gs}$ for any algebraically closed field Ω/k. Moreover, we give a modular interpretation of this decomposition, as the domains of the morphisms f^{rs}_{Σ,u^b} can also be described in terms of moduli spaces of so-called (Σ, u^b)-equivariant representations of Q by Corollary 4.34. For the trivial modifying family $u = 1$, the domain of f_Σ can also be described as a moduli space for representations of a quotient quiver Q/Σ (*cf.* Definition 4.4 and Corollary 4.6).

Let us explain the connections between the present paper and previous work on quiver automorphism groups. Contravariant involutions of a quiver were studied by Derksen and Weyman [5] and later by Young [35], where Young's motivation comes from physics and as an application he constructs orientifold Donaldson-Thomas invariants. In [35], the action of a contravariant involution is also modified using what is called a 'duality structure', which corresponds to our notion of modifying families. Motivated by questions in representation theory, Henderson and Licata study actions of so-called 'admissible' covariant automorphisms on Nakajima quiver varieties of type A and prove a decomposition of the fixed locus [13].

Many of our techniques can be applied to study fixed loci of group actions on more general GIT quotients. However, for the results involving group cohomology, one would need to assume that the stabiliser group of all GIT stable points is a fixed subgroup (analogous to the fixed subgroup $\Delta \subset \mathbf{G}_{Q,d}$), and one would need this fixed subgroup to be Abelian, in order to define the type map, as the second cohomology is only defined for an Abelian coefficient group. Over the complex numbers, by the Kempf-Ness Theorem, one can relate GIT quotients with symplectic reductions. In the symplectic setting, by using moment maps, one could also perform an analogous study of such fixed loci.

One can construct an algebraic symplectic analogue of $\mathcal{M}^{\theta-ss}_{Q,d}$ as a moduli space $\mathcal{N}_{\overline{Q}}$ of representations of a doubled quiver $\overline{Q}$ modulo relations defined by a moment map. Over the complex numbers, if $\mathcal{N}_{\overline{Q}}$ is smooth, it is hyperkähler; for example, Nakajima quiver varieties can be described in this way. In §5, we construct submanifolds of algebraic symplectic (and hyperkähler, when $k = \mathbb{C}$) quiver varieties as fixed loci for actions of subgroups of $\mathrm{Aut}(Q)$ (and for complex conjugation, when $k = \mathbb{C}$) with rich symplectic (and holomorphic) geometry. Such submanifolds can be described in the language of branes [16], which will be recalled in Section 5. The study of branes in Nakajima quiver varieties was initiated in [6], where involutions such as complex conjugation, multiplication by -1 and transposition are used to construct branes in Nakajima quiver varieties. We add to this picture branes arising from actions of quiver automorphism groups. Interestingly, we construct hyperholomorphic branes (BBB-branes) in Theorem 5.10 from subgroups of $\mathrm{Aut}(\overline{Q})$ consisting of $\overline{Q}$-symplectic transformations (this is a combinatorial notion, *cf.* Definition 5.4), that need not be of order 2. In Theorem 5.11, we show that one can construct BAA-branes in $\mathcal{N}_{\overline{Q}}$ as fixed loci of $\overline{Q}$-anti-symplectic quiver involutions. By combining (anti)-symplectic quiver involutions with complex conjugation, we produce all possible brane types (*cf.* Corollary 5.12). For moduli spaces of principal G-Higgs bundles (G being a complex reductive group) over a smooth complex projective curve X, actions induced by involutions (or automorphisms) of X and involutions of G have been studied in [1, 2, 3, 7, 12, 8, 29, 30, 31]; the approach in these papers is usually based on the gauge-theoretic, rather than algebraic, constructions of those moduli spaces.

The structure of this paper is as follows. In §2, we explain how to construct moduli spaces of representations of a quiver over an arbitrary field k following King

[19], and we examine how (semi)stability behaves under base field extension. In §3, we study actions of Gal_k for a perfect field k, and in §4, we study actions of quiver automorphism groups. In §5, we apply our results to construct branes in hyperkähler quiver varieties, and in §6, we study some examples.

Notation. For a scheme S over a field k and a field extension L/k, we denote by S_L the base change of S to L. For a point $s \in S$, we let $\kappa(s)$ denote the residue field of s. A quiver $Q = (V, A, h, t)$ is an oriented graph, consisting of a finite vertex set V, a finite arrow set A, a tail map $t : V \longrightarrow A$ and a head map $h : A \longrightarrow V$.

2. Quiver representations over a field

Let $Q = (V, A, h, t)$ be a quiver and let k be a field.

Definition 2.1 (k-representation of Q)**.** A representation of Q in the category of k-vector spaces (or k-representation of Q) is a tuple $W := ((W_v)_{v\in V}, (\varphi_a)_{a\in A})$ where:

- W_v is a finite-dimensional k-vector space for all $v \in V$;
- $\varphi_a : W_{t(a)} \longrightarrow W_{h(a)}$ is a k-linear map for all $a \in A$.

There are natural notions of morphisms of quiver representations and subrepresentations. The dimension vector of a k-representation W is the tuple $d = (\dim_k W_v)_{v\in V}$; we then say W is d-dimensional.

2.1. Slope semistability.

Following King's construction of moduli spaces of quiver representations over an algebraically closed field [19], we introduce a stability parameter $\theta := (\theta_v)_{v\in V} \in \mathbb{Z}^V$ and the associated slope function μ_θ, defined for all non-zero k-representations W of Q, by

$$\mu_\theta(W) := \mu_\theta^k(W) := \frac{\sum_{v\in V} \theta_v \dim_k W_v}{\sum_{v\in V} \dim_k W_v} \in \mathbb{Q}.$$

Definition 2.2 (Semistability and stability)**.** A k-representation W of Q is:

(1) θ-semistable if $\mu_\theta(W') \leq \mu_\theta(W)$ for all k-subrepresentation $0 \neq W' \subset W$.
(2) θ-stable if $\mu_\theta(W') < \mu_\theta(W)$ for all k-subrepresentation $0 \neq W' \subsetneq W$.
(3) θ-polystable if it is isomorphic to a direct sum of θ-stable representations of equal slope.

The category of θ-semistable k-representations of Q with fixed slope $\mu \in \mathbb{Q}$ is an Abelian, Noetherian and Artinian category, so it admits Jordan-Hölder filtrations. The simple (resp. semisimple) objects in this category are precisely the stable (resp. polystable) representations of slope μ (proofs of these facts are readily obtained by adapting the arguments of [33] to the quiver setting). The graded object associated to any Jordan-Hölder filtration of a semistable representation is by definition polystable and its isomorphism class as a graded object is independent of the choice of the filtration. Two θ-semistable k-representations of Q are called S-equivalent if their associated graded objects are isomorphic.

Definition 2.3 (*Scss* subrepresentation)**.** Let W be a k-representation of k; then a k-subrepresentation $U \subset W$ is said to be strongly contradicting semistability (*scss*) with respect to θ if its slope is maximal among the slopes of all subrepresentations of W and, for any $W' \subset W$ with this property, we have $U \subset W' \Rightarrow U = W'$.

For a proof of the existence and uniqueness of the *scss* subrepresentation, we refer to [27, Lemma 4.4]. The *scss* subrepresentation satisfies $\mathrm{Hom}(U; W/U) = 0$. Using the existence and uniqueness of the *scss*, one can inductively construct a unique Harder–Narasimhan filtration with respect to θ; for example, see [27, Lemma 4.7].

We now turn to the study of how the notions of semistability and stability behave under a field extension L/k. A k-representation $W = ((W_v)_{v\in V}, (\varphi_a)_{a\in A})$ of Q determines an L-representation $L\otimes_k W := ((L\otimes_k W_v)_{v\in V}, (Id_L \otimes \varphi_a)_{a\in A})$ (or simply $L\otimes W$), where $L\otimes_k W_v$ is equipped with its canonical structure of L-vector space and $Id_L \otimes \varphi_a$ is the extension of the k-linear map φ_a by L-linearity. Note that the dimension vector of $L\otimes_k W$ as an L-representation is the same as the dimension vector of W as a k-representation. We prove that semistability of quiver representations is invariant under base field extension, by following the proof of the analogous statement for sheaves given in [21, Proposition 3] and [14, Theorem 1.3.7].

Proposition 2.4. *Let L/k be a field extension and let W be a k-representation. For a stability parameter $\theta \in \mathbb{Z}^V$, the following statements hold.*

1. *If $L\otimes_k W$ is θ-semistable (resp. θ-stable) as an L-representation, then W is θ-semistable (resp. θ-stable) as a k-representation.*
2. *If W is θ-semistable as a k-representation, then $L\otimes_k W$ is θ-semistable as an L-representation.*

Moreover, if $(W^i)_{1\le i\le l}$ is the Harder-Narasimhan filtration of W, then $(L\otimes_k W^i)_{1\le i\le l}$ is the Harder-Narasimhan filtration of $L\otimes_k W$.

Proof. Let us suppose that $L\otimes_k W$ is θ-semistable as an L-representation. Then, given a k-subrepresentation $W' \subset W$, we have

$$\mu_\theta^k(W') = \mu_\theta^L(L\otimes_k W') \le \mu_\theta^L(L\otimes_k W) = \mu_\theta^k(W).$$

Therefore, W is necessarily θ-semistable as a k-representation. The proof shows that (1) also holds for stability.

First we can reduce (2) to finitely generated extensions L/k as follows. Let W^L be an L-subrepresentation of $L\otimes_k W$. For each $v\in V$, choose an L-basis $(b_j^v)_{1\le j\le d_v}$ of W_v^L and write $b_j^v = \sum_i a_{ij}^v e_{ij}^v$ (a finite sum with $a_{ij}^v \in L$ and $e_{ij}^v \in W_v$). Let L' be the subfield of L generated by the a_{ij}^v. The (e_{ij}^v) generate an L'-subrepresentation $W^{L'}$ of $L'\otimes_k W$ that satisfies $L\otimes_{L'} W^{L'} = W^{L'}$. If $L\otimes_k W$ is not semistable, there exists $W^L \subset L\otimes_k W$ such that $\mu_\theta^L(W^L) > \mu_\theta^L(L\otimes_k W) = \mu_\theta(W)$, then $L'\otimes_k W$ is not semistable, as $\mu^{L'}(W^{L'}) > \mu_\theta^{L'}(L'\otimes_k W)$.

By filtering L/k by various subfields, it suffices to verify the following cases:

1. L/k is a Galois extension;
2. L/k is a separable algebraic extension;
3. L/k is a purely inseparable finite extension;
4. L/k is a purely transcendental extension, of transcendence degree 1.

For i), we prove the statement by contrapositive, using the existence and uniqueness of the *scss* $U \subsetneq L\otimes_k W$ with $\mu_\theta^L(U) > \mu_\theta^L(L\otimes_k W)$. For $\tau \in \mathrm{Aut}(L/k)$, we construct a L-subrepresentation $\tau(U)$ of $L\otimes_k W$ of the same dimension vector and slope as U as follows. For each $v\in V$, the k-automorphism τ of L induces an L-semilinear transformation of $L\otimes_k W_v$ (i.e., an additive map satisfying $\tau(zw) = \tau(z)\tau(w)$ for all $z\in L$ and all $w\in W_v$), which implies that $\tau(U_v)$ is an L-vector subspace of $L\otimes_k W_v$, and the map $\tau\varphi_a\tau^{-1} : \tau(U_{t(a)}) \longrightarrow \tau(U_{h(a)})$ is

L-linear. By uniqueness of the *scss* subrepresentation U, we must have $\tau(U) = U$ for all $\tau \in \mathrm{Aut}(L/k)$. Moreover, for all $v \in V$, the k-vector space $U_v^{\mathrm{Aut}(L/k)}$ is a subspace of $(L \otimes_k W_v)^{\mathrm{Aut}(L/k)} = W_v$, as L/k is Galois. Then $U^{\mathrm{Aut}(L/k)} \subset W$ is a k-subrepresentation with $\mu_\theta(U^{\mathrm{Aut}(L/k)}) = \mu_\theta^L(U)$; thus W is not semistable.

In case ii), we choose a Galois extension N of k containing L; then we can conclude the claim using i) and Part (1).

For iii), by Jacobson descent, an L-subrepresentation $W^L \subset L \otimes_k W$ descends to a k-subrepresentation of W if and only if W^L is invariant under the algebra of k-derivations of L, which is the case for the *scss* L-subrepresentation $U \subset L \otimes_k W$. Indeed, let us consider a derivation $\delta \in \mathrm{Der}_k(L)$ and, for all $v \in V$, the induced transformation $(\psi_\delta)_v := (\delta \otimes_k \mathrm{Id}_{W_v}) : L \otimes_k W_v \longrightarrow L \otimes_k W_v$. Then, for all $v \in V$, all $\lambda \in L$ and all $u \in U_v$, one has $(\psi_\delta)_v(\lambda u) = \delta(\lambda)u + \lambda\psi_\delta(u)$. As the composition

$$\overline{(\psi_\delta)_v} : U_v \hookrightarrow L \otimes_k W_v \stackrel{(\psi_\delta)_v}{\longrightarrow} L \otimes_k W_v \longrightarrow (L \otimes_k W_v)/U_v$$

is L-linear, we obtain a morphism of L-representations $\overline{\psi_\delta} : U \longrightarrow (L \otimes_k W)/U$, which must be zero as U is the *scss* subrepresentation of $L \otimes_k W$. As U is invariant under ψ_δ, it descends to a k-subrepresentation of W; then we argue as in i).

For iv), we distinguish two cases. If k is infinite, the fixed subfield of $k(X)$ for the action of $\mathrm{Aut}(k(X)/k) \simeq \mathbf{PGL}(2;k)$ is k (*cf.* [28, p. 254]), so we can argue as in i). If k is finite, the fixed subfield of $k(X)$ for the action of $\mathrm{Aut}(k(X)/k)$ is strictly larger than k, as $\mathbf{PGL}(2;k)$ is finite. Let $\overline{k}$ be an algebraic closure of k. If W is semistable, so is $\overline{k} \otimes_k W$ in view of the above, since $\overline{k}/k$ is algebraic. As $\overline{k}$ is infinite, $\overline{k}(X) \otimes_{\overline{k}} (\overline{k} \otimes_k W) = \overline{k}(X) \otimes_k W$ is also semistable. Since $\overline{k}(X) \otimes_k W = \overline{k}(X) \otimes_{k(X)} (k(X) \otimes_k W)$, we can conclude that $k(X) \otimes_k W$ is a semistable $k(X)$-representation by Part (1). □

Remark 2.5. Part (2) of Proposition 2.4 is not true if we replace semistability by stability, as is evident if we set $k = \mathbb{R}$ and $L = \mathbb{C}$: for a θ-stable $\mathbb{R}$-representation W, its complexification $\mathbb{C} \otimes W$ is a θ-semistable $\mathbb{C}$-representation by Proposition 2.4 and either, for all $\mathbb{C}$-subrepresentations $U \subset \mathbb{C} \otimes W$, one has $\mu_\theta^{\mathbb{C}}(U) < \mu_\theta^{\mathbb{C}}(\mathbb{C} \otimes W)$, in which case $\mathbb{C} \otimes W$ is actually θ-stable as a $\mathbb{C}$-representation; or there exists a $\mathbb{C}$-subrepresentation $U \subset L \otimes W$ such that $\mu_\theta^{\mathbb{C}}(U) = \mu_\theta^{\mathbb{C}}(\mathbb{C} \otimes W)$. In the second case, let $\overline{U}$ be the $\mathbb{C}$-subrepresentation of $\mathbb{C} \otimes W$ obtained by applying the non-trivial element of $\mathrm{Aut}(\mathbb{C}/\mathbb{R})$ to U. Note that $\overline{U} \neq U$, as otherwise it would contradict the θ-stability of W as an $\mathbb{R}$-representation (as in the proof of Part (2) of Proposition 2.4). It is then not difficult, adapting the arguments of [26], to show that U is a θ-stable $\mathbb{C}$-representation and that $\mathbb{C} \otimes W \simeq U \oplus \overline{U}$; thus $\mathbb{C} \otimes W$ is only θ-polystable as a $\mathbb{C}$-representation.

This observation motivates the following definition.

Definition 2.6 (Geometric stability)**.** A k-representation W is θ-geometrically stable if $L \otimes_k W$ is θ-stable as an L-representation for all extensions L/k.

Evidently, the notion of geometric stability is invariant under field extension. It turns out that, if $k = \overline{k}$, then being geometrically stable is the same as being stable: this can be proved directly, as in [14, Corollary 1.5.11], or as a consequence of Theorem 2.11 below. In particular, this implies that a k-representation W is θ-geometrically stable if and only if $\overline{k} \otimes_k W$ is θ-stable (the proof is the same as in the final part of Proposition 2.4).

2.2. **Families of quiver representations.** A family of k-representations of Q parametrised by a k-scheme B is a representation of Q in the category of vector bundles over B/k, denoted $\mathcal{E} = ((\mathcal{E}_v)_{v\in V}, (\varphi_a)_{a\in A}) \longrightarrow B$. For $d = (d_v)_{v\in V} \in \mathbb{N}^V$, we say a family $\mathcal{E} \longrightarrow B$ is d-dimensional if, for all $v \in V$, the rank of $\mathcal{E}_v$ is d_v. If $f : B' \longrightarrow B$ is a morphism of k-schemes, there is a pullback family $f^*\mathcal{E} := (f^*\mathcal{E}_v)_{v\in V}$ over B'. For $b \in B$ with residue field $\kappa(b)$, we let $\mathcal{E}_b$ denote the $\kappa(b)$-representation obtained by pulling back $\mathcal{E}$ along $u_b : \operatorname{Spec} \kappa(b) \longrightarrow B$.

Definition 2.7 (Semistability in families)**.** A family $\mathcal{E} \longrightarrow B$ of k-representations of Q is called:

(1) θ-semistable if, for all $b \in B$, the $\kappa(b)$-representation $\mathcal{E}_b$ is θ-semistable.
(2) θ-geometrically stable if, for all $b \in B$, the $\kappa(b)$-representation $\mathcal{E}_b$ is θ-geometrically stable.

For a family $\mathcal{E} \longrightarrow B$ of k-representations of Q, the subset of points $b \in B$ for which $\mathcal{E}_b$ is θ-semistable (resp. θ-geometrically stable) is open; one can prove this by adapting the argument in [14, Proposition 2.3.1]. By Proposition 2.4 and Definition 2.6, the pullback of a θ-semistable (resp. θ-geometrically stable) family is semistable (resp. geometrically stable). Therefore, we can introduce the following moduli functors:

$$F_{Q,d}^{\theta-ss} : (Sch_k)^{\mathrm{op}} \longrightarrow \mathrm{Sets} \quad \text{and} \quad F_{Q,d}^{\theta-ss} : (Sch_k)^{\mathrm{op}} \longrightarrow \mathrm{Sets}, \tag{2.1}$$

where $(Sch_k)^{\mathrm{op}}$ denotes the opposite category of the category of k-schemes and, for $B \in Sch_k$, we have that $F_{Q,d}^{\theta-ss}(B)$ (resp. $F_{Q,d}^{\theta-gs}(B)$) is the set of isomorphism classes of θ-semistable (resp. θ-geometrically stable) d-dimensional families over B of k-representations of Q.

We follow the convention that a coarse moduli space for a moduli functor $F : (Sch_k)^{\mathrm{op}} \longrightarrow \mathrm{Sets}$ is a scheme $\mathcal{M}$ that corepresents F (that is, there is a natural transformation $F \longrightarrow \mathrm{Hom}(-, \mathcal{M})$, which is universal).

2.3. **The GIT construction of the moduli space.** We fix a ground field k and dimension vector $d = (d_v)_{v\in V} \in \mathbb{N}^V$; then every d-dimensional k-representation of Q is isomorphic to a point of the following affine space over k

$$\mathrm{Rep}_{Q,d} := \prod_{a\in A} \mathrm{Mat}_{d_{h(a)}\times d_{t(a)}}.$$

The reductive group $\mathbf{G}_{Q,d} := \prod_{v\in V} \mathbf{GL}_{d_v}$ over k acts algebraically on $\mathrm{Rep}_{Q,d}$ by conjugation: for $g = (g_v)_{v\in V} \in \mathbf{G}_{Q,d}$ and $M = (M_a)_{a\in A} \in \mathrm{Rep}_{Q,d}$, we have

$$g \cdot M := (g_{h(a)} M_a g_{t(a)}^{-1})_{a\in A}. \tag{2.2}$$

There is a tautological family $\mathcal{F} \longrightarrow \mathrm{Rep}_{Q,d}$ of d-dimensional k-representations of Q, where $\mathcal{F}_v$ is the trivial rank d_v vector bundle on $\mathrm{Rep}_{Q,d}$.

Lemma 2.8. *The tautological family $\mathcal{F} \longrightarrow \mathrm{Rep}_{Q,d}$ has the local universal property; that is, for every family $\mathcal{E} = ((\mathcal{E}_v)_{v\in V}, (\varphi_a)_{a\in A}) \longrightarrow B$ of representations of Q over a k-scheme B, there is an open covering $B = \cup_{i\in I} B_i$ and morphisms $f_i : B_i \longrightarrow \mathrm{Rep}_{Q,d}$ such that $\mathcal{E}|_{B_i} \cong f_i^*\mathcal{F}$.*

Proof. Take an open cover of B on which all the (finitely many) vector bundles $\mathcal{E}_v$ are trivialisable, then the morphisms f_i are determined by the morphisms φ_a. □

We will construct a quotient of the $\mathbf{G}_{Q,d}$-action on $\mathrm{Rep}_{Q,d}$ via geometric invariant theory (GIT) using a linearisation of the action by a stability parameter $\theta = (\theta_v)_{v\in V} \in \mathbb{Z}^V$. Let us set $\theta' := (\theta'_v)_{v\in V}$ where $\theta'_v := \theta_v \sum_{\alpha\in V} d_\alpha - \sum_{\alpha\in V} \theta_\alpha d_\alpha$ for all $v \in V$; then one can easily check that θ'-(semi)stability is equivalent to θ-(semi)stability. We define a character $\chi_\theta : \mathbf{G}_{Q,d} \longrightarrow \mathbb{G}_m$ by

$$\chi_\theta((g_v)_{v\in V}) := \prod_{v\in V} (\det g_v)^{-\theta'_v}. \tag{2.3}$$

Any such character $\chi : \mathbf{G}_{Q,d} \longrightarrow \mathbb{G}_m$ defines a lifting of the $\mathbf{G}_{Q,d}$-action on $\mathrm{Rep}_{Q,d}$ to the trivial line bundle $\mathrm{Rep}_{Q,d} \times \mathbb{A}^1$, where $\mathbf{G}_{Q,d}$ acts on $\mathbb{A}^1$ via multiplication by χ. As the subgroup $\Delta \subset \mathbf{G}_{Q,d}$, whose set of R-points (for R a k-algebra) is

$$\Delta(R) := \{(tI_{d_v})_{v\in V} : t \in R^\times\} \cong \mathbb{G}_m(R), \tag{2.4}$$

acts trivially on $\mathrm{Rep}_{Q,d}$, invariant sections only exist if $\chi^{(R)}(\Delta(R)) = \{1_{R^\times}\}$ for all R; this holds for χ_θ, as $\sum_{v\in V} \theta'_v d_v = 0$. Let $\mathcal{L}_\theta$ denote the line bundle $\mathrm{Rep}_{Q,d} \times \mathbb{A}^1$ endowed with the $\mathbf{G}_{Q,d}$-action induced by χ_θ and by $\mathcal{L}_\theta^n$ its n-th tensor power for $n \geq 1$ (endowed with the action of χ_θ^n). The invariant sections of $\mathcal{L}_\theta^n$ are morphisms $f : \mathrm{Rep}_{Q,d} \longrightarrow \mathbb{A}^1$ satisfying $f(g \cdot M) = \chi_\theta(g)^n f(M)$, for all $g \in \mathbf{G}_{Q,d}$ and all $M \in \mathrm{Rep}_{Q,d}$.

Definition 2.9 (GIT (semi)stability)**.** A point $M \in \mathrm{Rep}_{Q,d}$ is called:

1. χ_θ-semistable if there exists an integer $n > 0$ and a $\mathbf{G}_{Q,d}$-invariant section f of $\mathcal{L}_\theta^n$ such that $f(M) \neq 0$.
2. χ_θ-stable if there exists an integer $n > 0$ and a $\mathbf{G}_{Q,d}$-invariant section f of $\mathcal{L}_\theta^n$ such that $f(M) \neq 0$, the action of $\mathbf{G}_{Q,d}$ on $(\mathrm{Rep}_{Q,d})_f$ is closed and $\dim_{\kappa(M)}(\mathrm{Stab}(M)/\Delta_{\kappa(M)}) = 0$, where $\mathrm{Stab}(M) \subset \mathbf{G}_{Q,d,\kappa(M)}$ is the stabiliser group scheme of M.

The set of χ_θ-(semi)stable points in $\mathrm{Rep}_{Q,d}$ is denoted $\mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s}$ or, if we wish to emphasise the group, $\mathrm{Rep}_{Q,d}^{(\mathbf{G}_{Q,d},\chi_\theta)-(s)s}$.

Evidently, $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}$ and $\mathrm{Rep}_{Q,d}^{\chi_\theta-s}$ are $\mathbf{G}_{Q,d}$-invariant open subsets. Moreover, these subsets commute with base change (*cf.* [23, Proposition 1.14] and [34, Lemma 2]). Mumford's GIT (or, more precisely, Seshadri's extension of GIT [34]) provides a categorical and good quotient of the $\mathbf{G}_{Q,d}$-action on $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}$

$$\pi : \mathrm{Rep}_{Q,d}^{\chi_\theta-ss} \longrightarrow \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d} := \mathrm{Proj} \bigoplus_{n\geq 0} H^0(\mathrm{Rep}_{Q,d}, \mathcal{L}_\theta^n)^{\mathbf{G}_{Q,d}},$$

which restricts to a geometric quotient $\pi|_{\mathrm{Rep}_{Q,d}^{\chi_\theta-s}} : \mathrm{Rep}_{Q,d}^{\chi_\theta-s} \longrightarrow \mathrm{Rep}_{Q,d}^{\chi_\theta-s} / \mathbf{G}_{Q,d}$.

Given a geometric point $M : \mathrm{Spec}\,\Omega \longrightarrow \mathrm{Rep}_{Q,d}$, let us denote by $\Lambda(M)$ the set of 1-parameter subgroups $\lambda : \mathbb{G}_{m,\Omega} \longrightarrow \mathbf{G}_{Q,d,\Omega}$ such that the morphism $\mathbb{G}_{m,\Omega} \longrightarrow \mathrm{Rep}_{Q,d,\Omega}$, given by the λ-action on M, extends to $\mathbb{A}^1_\Omega$. As $\mathrm{Rep}_{Q,d}$ is separated, if this morphism extends, its extension is unique. If M_0 denotes the image of $0 \in \mathbb{A}^1_\Omega$, the weight of the induced action of $\mathbb{G}_{m,\Omega}$ on $\mathcal{L}_{\theta,\Omega}|_{M_0}$ is $(\chi_{\theta,\Omega}, \lambda) \in \mathbb{Z}$, where $(-,-)$ denotes the natural pairing of characters and 1-parameter subgroups.

Proposition 2.10 (Hilbert-Mumford criterion [19])**.** *For a geometric point $M : \mathrm{Spec}\,\Omega \longrightarrow \mathrm{Rep}_{Q,d}$, we have:*

1. *M is χ_θ-semistable if and only if $(\chi_{\theta,\Omega}, \lambda) \geq 0$ for all $\lambda \in \Lambda(M)$;*

(2) *M is χ_θ-stable if and only if $(\chi_{\theta,\Omega},\lambda)\geq 0$ for all $\lambda\in\Lambda(M)$, and $(\chi_{\theta,\Omega},\lambda)=0$ implies $\operatorname{Im}\lambda\subset\operatorname{Stab}(M)$, where $\operatorname{Stab}(M)\subset\mathbf{G}_{Q,d,\Omega}$ is the stabiliser group scheme of M.*

Proof. If k is algebraically closed and $\Omega=k$, this is [19, Proposition 2.5]; then the above result follows as GIT (semi)stability commutes with base change. □

Before we relate slope (semi)stability and GIT (semi)stability for quiver representations, let $\operatorname{Rep}_{Q,d}^{\theta-ss}$ (resp. $\operatorname{Rep}_{Q,d}^{\theta-gs}$) be the open subset of points in $\operatorname{Rep}_{Q,d}$ over which the tautological family $\mathcal{F}$ is θ-semistable (resp. θ-geometrically stable).

Proposition 2.11. *For $\theta\in\mathbb{Z}^V$, we have the following equalities of k-schemes:*
(1) $\operatorname{Rep}_{Q,d}^{\theta-ss}=\operatorname{Rep}_{Q,d}^{\chi_\theta-ss}$;
(2) $\operatorname{Rep}_{Q,d}^{\theta-gs}=\operatorname{Rep}_{Q,d}^{\chi_\theta-s}$.

Proof. Since all of these k-subschemes of $\operatorname{Rep}_{Q,d}$ are open, it suffices to verify these equalities on $\overline{k}$-points, for which one uses [19, Proposition 3.1] (we note that we use the opposite inequality to King in our definition of slope (semi)stability, but this is rectified by the minus sign appearing in (2.3) for the definition of χ_θ). □

Proposition 2.11 implies the result claimed at the end of §2.1: a k-representation W is θ-geometrically stable if and only if $\overline{k}\otimes_k W$ is θ-stable (in particular, if $k=\overline{k}$, then θ-geometric stability is equivalent to θ-stability). Finally we give the existence of coarse moduli spaces of θ-semistable (resp. θ-geometrically stable) k-representations of Q for an arbitrary field k. For an algebraically closed field k, this result is proved in [19, Proposition 5.2].

Corollary 2.12. *The k-variety $\mathcal{M}_{Q,d}^{\theta-ss}:=\operatorname{Rep}_{Q,d}/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ is a coarse moduli space for the functor $F_{Q,d}^{\theta-ss}$ and the natural map $F_{Q,d}^{\theta-ss}(\overline{k})\longrightarrow\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ is surjective. Moreover, $\mathcal{M}_{Q,d}^{\theta-gs}:=\operatorname{Rep}_{Q,d}^{\chi_\theta-s}/\mathbf{G}_{Q,d}$ is an open k-subvariety of $\mathcal{M}_{Q,d}^{\theta-ss}$ which is a coarse moduli space for the functor $F_{Q,d}^{\theta-gs}$ and the natural map $F_{Q,d}^{\theta-gs}(\overline{k})\longrightarrow\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ is bijective.*

Proof. First, we verify that $\mathcal{M}_{Q,d}^{\theta-ss}$ is a k-variety: it is of finite type over k, as the ring of sections of powers of $\mathcal{L}_\theta$ that are invariant for the reductive group $\mathbf{G}_{Q,d}$ is finitely generated. Moreover, $\mathcal{M}_{Q,d}^{\theta-ss}$ is separated, as it is projective over the affine k-scheme $\operatorname{Spec}\mathcal{O}(\operatorname{Rep}_{Q,d})^{\mathbf{G}_{Q,d}}$. Finally $\mathcal{M}_{Q,d}^{\theta-ss}$ is integral, as $\operatorname{Rep}_{Q,d}^{\chi_\theta-ss}$ is and this property is inherited by the categorical quotient.

Since the tautological family $\mathcal{F}^{\theta-ss}\longrightarrow\operatorname{Rep}_{Q,d}^{\theta-ss}$ has the local universal property by Lemma 2.8 and also the $\mathbf{G}_{Q,d}$-action on $\operatorname{Rep}_{Q,d}$ is such that $M,M'\in\operatorname{Rep}_{Q,d}$ lie in the same $\mathbf{G}_{Q,d}$-orbit if and only if $\mathcal{F}_M\cong\mathcal{F}_{M'}$, it follows that any $\mathbf{G}_{Q,d}$-invariant morphism $p:\operatorname{Rep}_{Q,d}^{\theta-ss}\longrightarrow Y$ is equivalent to a natural transformation $\eta_p:F_{Q,d}^{\theta-ss}\longrightarrow\operatorname{Hom}(-,Y)$ (*cf.* [24, Proposition 2.13]). As $\operatorname{Rep}_{Q,d}^{\theta-ss}=\operatorname{Rep}_{Q,d}^{\chi_\theta-ss}$ by Proposition 2.11, and as $\pi:\operatorname{Rep}_{Q,d}^{\chi_\theta-ss}\longrightarrow\operatorname{Rep}_{Q,d}/\!/_{\chi_\theta}\mathbf{G}_{Q,d}=\mathcal{M}_{Q,d}^{\theta-ss}$ is a universal $\mathbf{G}_{Q,d}$-invariant morphism, it follows that $\mathcal{M}_{Q,d}^{\theta-ss}$ co-represents $F_{Q,d}^{\theta-ss}$, and similarly $\mathcal{M}_{Q,d}^{\theta-gs}$ co-represents $F_{Q,d}^{\theta-gs}$.

The points of $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ are in bijection with equivalence classes of $\mathbf{G}_{Q,d}(\overline{k})$-orbits of χ_θ-semistable $\overline{k}$-points, where $\overline{k}$-points M_1 and M_2 are equivalent if their orbit closures intersect in $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$ (*cf.* [34, Theorem 4]). By [19, Proposition 3.2.(ii)], this is the same as the S-equivalence of $\mathcal{F}_{M_1}$ and $\mathcal{F}_{M_2}$ as θ-semistable $\overline{k}$-representation of Q; hence the surjectivity of the natural map $F_{Q,d}^{\theta-ss}(\overline{k}) \longrightarrow \mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$. Likewise, $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ is in bijection with the set of $\mathbf{G}_{Q,d}(\overline{k})$-orbits of χ_θ-stable $\overline{k}$-points of $\mathrm{Rep}_{Q,d}$, which, by [19, Proposition 3.1], is in bijection with the set of θ-stable d-dimensional $\overline{k}$-representations of Q. □

We end this section with a result that is used repeatedly in Sections 3 and 4.

Corollary 2.13. *For* $M \in \mathrm{Rep}_{Q,d}^{\theta-gs}$, *we have* $\mathrm{Stab}(M) = \Delta_{\kappa(M)} \subset \mathbf{G}_{Q,d,\kappa(M)}$.

Proof. $\mathrm{Stab}(M) \subset \mathbf{G}_{Q,d,\kappa(M)}$ is isomorphic to $\mathrm{Aut}(\mathcal{F}_M)$, where $\mathcal{F} \longrightarrow \mathrm{Rep}_{Q,d}$ is the tautological family, and $\mathcal{F}_M$ is θ-geometrically stable. The endomorphism group of a stable k-representation of Q is a finite dimensional division algebra over k (*cf.* [14, Proposition 1.2.8]). Let $\overline{\kappa(M)}$ be an algebraic closure of $\kappa(M)$; then, as $\overline{\kappa(M)} \otimes \mathcal{F}_M$ is θ-stable and $\overline{\kappa(M)}$ is algebraically closed, $\mathrm{End}(\overline{\kappa(M)} \otimes \mathcal{F}_M) = \overline{\kappa(M)}$. Since $\overline{\kappa(M)} \otimes \mathrm{End}(\mathcal{F}_M) \subset \mathrm{End}(\overline{\kappa(M)} \otimes \mathcal{F}_M)$, it follows that $\mathrm{End}(\mathcal{F}_M) = \kappa(M)$ and thus $\mathrm{Aut}(\mathcal{F}_M) \simeq \Delta_{\kappa(M)}$. □

3. Galois actions

Throughout this section, we assume that k is a perfect field and we fix an algebraic closure $\overline{k}$ of k. For any k-scheme X, there is a left action of the Galois group $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}/k)$ on the set of $\overline{k}$-points $X(\overline{k})$ as follows: for $\tau \in \mathrm{Gal}_k$ and $x : \mathrm{Spec}\,\overline{k} \longrightarrow X$, we let $\tau \cdot x := x \circ \tau^*$, where $\tau^* : \mathrm{Spec}\,\overline{k} \longrightarrow \mathrm{Spec}\,\overline{k}$ is the morphism of k-scheme induced by the k-algebra homomorphism $\tau : \overline{k} \longrightarrow \overline{k}$. Since k is perfect, the fixed-point set the Gal_k-action on $X(\overline{k})$ is the set of k-points of X: $X(k) = X(\overline{k})^{\mathrm{Gal}_k}$. If $X_{\overline{k}} = \mathrm{Spec}\,\overline{k} \times_{\mathrm{Spec}\,k} X$, then $X_{\overline{k}}(\overline{k}) = X(\overline{k})$ and Gal_k acts on $X_{\overline{k}}$ by k-scheme automorphisms and, as k is perfect, we can recover X as $X_{\overline{k}}/\mathrm{Gal}_k$.

3.1. **Rational points of the moduli space.** The coarse moduli space $\mathcal{M}_{Q,d}^{\theta-ss}$ constructed in Section 2 is a k-variety and so the Galois group $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}/k)$ acts on $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ as described above and the fixed points of this action are the k-rational points. Alternatively, we can describe this action using the presentation of $\mathcal{M}_{Q,d}^{\theta-ss}$ as the GIT quotient $\mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}$. The Gal_k-action on $\mathrm{Rep}_{Q,d}(\overline{k}) = \prod_{a\in A} \mathrm{Mat}_{d_{h(a)}\times d_{t(a)}}(\overline{k})$ and $\mathbf{G}_{Q,d}(\overline{k}) = \prod_{v\in V} \mathbf{GL}_{d_v}(\overline{k})$ is given by applying a k-automorphism $\tau \in \mathrm{Gal}_k = \mathrm{Aut}(\overline{k}/k)$ to the entries of the matrices $(M_a)_{a\in A}$ and $(g_v)_{v\in V}$. Both actions are by homeomorphisms in the Zariski topology and the second action is by group automorphisms and preserves the subgroup $\Delta(\overline{k})$ defined in (2.4). We denote these actions as follows

$$\begin{array}{rccc} \Phi: & \mathrm{Gal}_k \times \mathrm{Rep}_{Q,d}(\overline{k}) & \longrightarrow & \mathrm{Rep}(\overline{k}) \\ & \big(\tau, (M_a)_{a\in A}\big) & \longmapsto & \big(\tau(M_a)\big)_{a\in A} \end{array} \tag{3.1}$$

and

$$\begin{array}{rccc} \Psi: & \mathrm{Gal}_k \times \mathbf{G}_{Q,d}(\overline{k}) & \longrightarrow & \mathbf{G}_{Q,d}(\overline{k}) \\ & \big(\tau, (g_v)_{v\in V}\big) & \longmapsto & \big(\tau(g_v)\big)_{v\in V}. \end{array} \tag{3.2}$$

They satisfy the following compatibility relation with the action of $\mathbf{G}_{Q,d}(\overline{k})$ on $\mathrm{Rep}_{Q,d}(\overline{k})$: for all $g \in \mathbf{G}_{Q,d}(\overline{k})$, all $M \in \mathrm{Rep}(\overline{k})$ and all $\tau \in \mathrm{Gal}_k$, one has

$$\Phi_\tau(g \cdot M) = \Psi_\tau(g) \cdot \Phi_\tau(M) \tag{3.3}$$

(i.e., the $\mathbf{G}_{Q,d}(\overline{k})$-action on $\mathrm{Rep}_{Q,d}(\overline{k})$ extends to an action of $\mathbf{G}_{Q,d}(\overline{k}) \rtimes \mathrm{Gal}_k$). For convenience, we will often simply denote $\Phi_\tau(M)$ by $\tau(M)$ and $\Psi_\tau(g)$ by $\tau(g)$.

To show that the action Φ preserves the semistable set $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$ with respect to the character χ_θ defined at (2.3), we will show that the action of Gal_k preserves the χ_θ-semi-invariant functions. By definition, $f : \mathrm{Rep}_{Q,d}(\overline{k}) \longrightarrow \overline{k}$ is a χ_θ-semi-invariant function if there exists $n > 0$ such that f is a $\mathbf{G}_{Q,d}(\overline{k})$-equivariant function for the $\mathbf{G}_{Q,d}(\overline{k})$-action on $\overline{k}$ given by χ_θ^n; i.e., $f(g \cdot M) = \chi_\theta^n(g) f(M)$ for all $g \in \mathbf{G}_{Q,d}(\overline{k})$ and $M \in \mathrm{Rep}_{Q,d}$. Since χ_θ is Gal_k-equivariant, we claim that, for any $\tau \in \mathrm{Gal}_k$ and any χ_θ-semi-invariant function f, the function

$$\begin{array}{rccc} \tau \cdot f : & \mathrm{Rep}_{Q,d}(\overline{k}) & \longrightarrow & \overline{k} \\ & M & \longmapsto & (\tau \circ f \circ \Phi_{\tau^{-1}})(M) \end{array}$$

is χ_θ-semi-invariant. Indeed, by the compatibility relation (3.3), we have, for all $\tau \in \mathrm{Gal}_k$, all $g \in \mathbf{G}_{Q,d}(\overline{k})$ and all $M \in \mathrm{Rep}(\overline{k})$,

$$\begin{aligned} (\tau \cdot f)(g \cdot M) = (\tau f \tau^{-1})(g \cdot M) &= \big(\tau \circ f\big)\big(\tau^{-1}(g) \cdot \tau^{-1}(M)\big) \\ &= \tau\big(\chi_\theta^n(\tau^{-1}(g)) f(\tau^{-1}(M))\big) \\ &= \tau\big(\tau^{-1}(\chi_\theta^n(g))\big)\tau\big(f(\tau^{-1}(M))\big) \\ &= \chi_\theta^n(g)\big((\tau \cdot f)(M)\big). \end{aligned}$$

Proposition 3.1. *The* Gal_k*-action on* $\mathrm{Rep}_{Q,d}(\overline{k})$ *preserves* $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$*. Moreover, if* M_1, M_2 *are two GIT-semistable points whose* $\mathbf{G}_{Q,d}(\overline{k})$*-orbits closures meet in* $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$*, then, for all* $\tau \in \mathrm{Gal}_k$*, the same is true for* $\tau(M_1)$ *and* $\tau(M_2)$*.*

Proof. Let $M \in \mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$; then there is a χ_θ-semi-invariant function f such that $f(M) \neq 0$. Then $(\tau \cdot f)(\tau(M)) = \tau(f(M)) \neq 0$ for all $\tau \in \mathrm{Gal}_k$, so $\tau(M)$ is GIT-semistable, as $\tau \cdot f$ is a χ_θ-semi-invariant function. The second statement follows from the compatibility relation (3.3) and the continuity of τ in the Zariski topology of $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$. □

The compatibility relation (3.3) also implies that, for $M \in \mathrm{Rep}_{Q,d}(\overline{k})$ and $\tau \in \mathrm{Gal}_k$, the stabiliser of $\tau(M)$ in $\mathbf{G}_{Q,d}(\overline{k})$ is $\mathrm{Stab}_{\mathbf{G}_{Q,d}(\overline{k})}\big(\tau(M)\big) = \tau\big(\mathrm{Stab}_{\mathbf{G}_{Q,d}(\overline{k})}(M)\big)$. In particular, if $\mathrm{Stab}_{\mathbf{G}_{Q,d}(\overline{k})}(M) = \Delta(\overline{k})$, the same holds for $\tau(M)$, which implies the following.

Proposition 3.2. *If* $M \in \mathrm{Rep}_{Q,d}(\overline{k})$ *is GIT-stable with respect to* χ_θ*, then so is* $\tau(M)$*.*

Proof. This follows from the above remarks and the definition of GIT stability, as the stabiliser of a GIT stable $\overline{k}$-point is equal to $\Delta(\overline{k})$ by Corollary 2.13. □

Propositions 3.1 and 3.2 combined with the compatibility relation (3.3) readily imply that Gal_k acts on the set of $\overline{k}$-points of the k-varieties $\mathcal{M}_{Q,d}^{\theta-ss} = \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}$ and $\mathcal{M}_{Q,d}^{\theta-gs} = \mathrm{Rep}_{Q,d}^{\theta-gs} / \mathbf{G}_{Q,d}$. Indeed, $(\mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d})(\overline{k})$ is the set of $\mathbf{G}_{Q,d}(\overline{k})$-orbits in $\mathrm{Rep}_{Q,d}^{ss}(\overline{k})$ modulo the equivalence relation $\mathcal{O}_{M_1} \sim \mathcal{O}_{M_2}$ if $\overline{\mathcal{O}_{M_1}} \cap \overline{\mathcal{O}_{M_2}} \neq \emptyset$ in $\mathrm{Rep}_{Q,d}^{\chi_\theta-ss}(\overline{k})$, and $(\mathrm{Rep}_{Q,d}^{\theta-gs} / \mathbf{G}_{Q,d})(\overline{k})$ is the orbit space $(\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k}) / \mathbf{G}_{Q,d}(\overline{k}))$, on which the Gal_k-action is given by

$$(\mathbf{G}_{Q,d}(\overline{k}) \cdot M) \longmapsto (\mathbf{G}_{Q,d}(\overline{k}) \cdot \tau(M)) \tag{3.4}$$

Since k is assumed to be a perfect field, this Gal_k-action on the $\overline{k}$-varieties $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ and $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ suffices to recover the k-schemes $\mathcal{M}_{Q,d}^{\theta-ss}$ and $\mathcal{M}_{Q,d}^{\theta-gs}$. In particular, the Gal_k-actions just described on $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})$ and $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ coincide with the ones described algebraically at the beginning of the present section.

To conclude this section, we give another description of the Galois action on $\mathrm{Rep}_{Q,d}(\overline{k})$ by intrinsically defining a Gal_k-action on arbitrary $\overline{k}$-representations of Q. If $W = ((W_v)_{v\in V},\ (\varphi_a)_{a\in A})$ is a $\overline{k}$-representation of Q, then, for $\tau \in \mathrm{Gal}_k$, we define W^τ to be the representation $(W_v^\tau, v\in V; \phi_a^\tau; a \in A)$ where:

- W_v^τ is the $\overline{k}$-vector space whose underlying Abelian group coincides with that of W_v and whose external multiplication is given by $\lambda \cdot_\tau w := \tau^{-1}(\lambda) w$ for $\lambda \in \overline{k}$ and $w \in W_v$.
- The map ϕ_a^τ coincides with ϕ_a, which is $\overline{k}$-linear for the new $\overline{k}$-vector space structures, as $\phi_a^\tau(\lambda \cdot_\tau w) = \phi_a(\tau^{-1}(\lambda)w) = \tau^{-1}(\lambda)\phi_a(w) = \lambda \cdot_\tau \phi_a^\tau(w)$.

If $\rho : W' \longrightarrow W$ is a morphism of $\overline{k}$-representations and $\tau \in \mathrm{Gal}_k$, we denote by $\rho^\tau : (W')^\tau \longrightarrow W^\tau$ the induced homomorphism (which set-theoretically coincides with ρ). With these conventions, we have a right action as $W^{\tau_1\tau_2} = (W^{\tau_1})^{\tau_2}$. Moreover, if we fix a $\overline{k}$-basis of each W_v, the matrix of ϕ_a^τ is $\tau(M_a)$, where M_a is the matrix of ϕ_a, so we recover the Gal_k-action (3.1). We note that the construction $W \longmapsto W^\tau$ is compatible with semistability and S-equivalence, thus showing in an intrinsic manner that Gal_k acts on the set of S-equivalence classes of semistable d-dimensional representations of Q.

3.2. **The natural map to the rational points.** By definition of the coarse moduli spaces $\mathcal{M}_{Q,d}^{\theta-ss}$ and $\mathcal{M}_{Q,d}^{\theta-gs}$, we have natural maps

$$F_{Q,d}^{\theta-ss}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-ss}(k) \quad \text{and} \quad F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k), \tag{3.5}$$

where $F_{Q,d}^{\theta-ss}$ and $F_{Q,d}^{\theta-gs}$ are the moduli functors defined at (2.1). As k is perfect, $\mathcal{M}_{Q,d}^{\theta-ss}(k) = \mathcal{M}_{Q,d}^{\theta-ss}(\overline{k})^{\mathrm{Gal}_k}$ and $\mathcal{M}_{Q,d}^{\theta-gs}(k) = \mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k}$. The goal of the present section is to use this basic fact in order to understand the natural maps (3.5). As a matter of fact, our techniques will only apply to $F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k)$, because $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ is the orbit space $\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})/\mathbf{G}_{Q,d}(\overline{k})$ and all GIT-stable points in $\mathrm{Rep}_{Q,d}(\overline{k})$ have the Abelian group $\Delta(\overline{k}) \simeq \overline{k}^\times$ as their stabiliser for the $\mathbf{G}_{Q,d}(\overline{k})$-action.

Note first that, by definition of the functor $F_{Q,d}^{\theta-gs}$, we have

$$F_{Q,d}^{\theta-gs}(k) \simeq \mathrm{Rep}_{Q,d}^{\theta-gs}(k)/\mathbf{G}_{Q,d}(k),$$

so the natural map $F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k)$ may be viewed as the map

$$\begin{array}{rccc} f_{\mathrm{Gal}_k} : & \mathrm{Rep}_{Q,d}^{\theta-gs}(k)/\mathbf{G}_{Q,d}(k) & \longrightarrow & \left(\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})/\mathbf{G}_{Q,d}(\overline{k})\right)^{\mathrm{Gal}_k} \\ & \mathbf{G}_{Q,d}(k)\cdot M & \longmapsto & \mathbf{G}_{Q,d}(\overline{k})\cdot(\overline{k}\otimes_k M) \end{array}$$

We will start by showing that f_{Gal_k} is injective. The proof is based on the following cohomological characterisation of the fibres of f_{Gal_k}, an analogue of which will be proved in Section 4 for group actions coming from automorphisms of Q.

Proposition 3.3. *The non-empty fibres of f_{Gal_k} are in bijection with the pointed set* $\ker\left(H^1(\mathrm{Gal}_k;\Delta(\overline{k})) \longrightarrow H^1(\mathrm{Gal}_k;\mathbf{G}_{Q,d}(\overline{k}))\right)$ *where this map is induced by the inclusion* $\Delta(\overline{k}) \subset \mathbf{G}_{Q,d}(\overline{k})$.

Before we prove this proposition, let us state and prove a corollary.

Corollary 3.4. *The natural map $F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k)$ is injective.*

Proof. Identify this map with f_{Gal_k}. Since $\Delta(\overline{k})$ is Gal_k-equivariantly isomorphic to $\mathbb{G}_m(\overline{k}) = \overline{k}^\times$, Hilbert's 90 shows that $H^1(\mathrm{Gal}_k; \Delta(\overline{k})) = \{1\}$ (see for instance [32, Proposition X.1.2 p.150], although technically in that reference the statement is proved for Galois groups of finite Galois extensions only, but the general case is obtained by the argument that $H^1(\varprojlim \mathrm{Gal}(L/k); \cdot\,) \simeq \varprojlim H^1(\mathrm{Gal}(L/k); \cdot\,)$, where the projective limit is taken over finite Galois sub-extensions L/k). Then Proposition 3.3 implies that f_{Gal_k} is injective. $\square$

The proof of Proposition 3.3 consists of setting up a map between a non-empty fibre of f_{Gal_k} and the kernel of the pointed map $H^1(\mathrm{Gal}_k; \Delta(\overline{k})) \longrightarrow H^1(\mathrm{Gal}_k; \mathbf{G}_{Q,d}(\overline{k}))$ and proving that it is a bijection. To define such a map, let us consider M_1, M_2 in $\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k}$ such that $\mathbf{G}_{Q,d}(\overline{k}) \cdot M_1 = \mathbf{G}_{Q,d}(\overline{k}) \cdot M_2$. Then there exists $g \in \mathbf{G}_{Q,d}(\overline{k})$ such that $g \cdot M_2 = M_1$. Therefore, for all $\tau \in \mathrm{Gal}_k$, we have

$$g^{-1} \cdot M_1 = M_2 = \tau(M_2) = \tau(g^{-1} \cdot M_1) = \tau(g^{-1}) \cdot \tau(M_1),$$

so $g\tau(g^{-1}) \in \mathrm{Stab}_{\mathbf{G}_{Q,d}(\overline{k})}(M_1) = \Delta(\overline{k})$.

Lemma 3.5. *Given M_1, M_2 in $\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k}$ and $g \in \mathbf{G}_{Q,d}(\overline{k})$ such that $g \cdot M_2 = M_1$, the map*

$$\begin{array}{rccc} \beta: & \mathrm{Gal}_k & \longrightarrow & \Delta(\overline{k}) \\ & \tau & \longmapsto & g\tau(g^{-1}) \end{array}$$

is a normalised $\Delta(\overline{k})$-valued 1-cocycle whose cohomology class only depends on the $\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k}$-orbits of M_1 and M_2. The cohomology class $[\beta]$ thus defined lies in the kernel of the pointed map $H^1(\mathrm{Gal}_k; \Delta(\overline{k})) \longrightarrow H^1(\mathrm{Gal}_k; \mathbf{G}_{Q,d}(\overline{k}))$ induced by the inclusion $\Delta(\overline{k}) \subset \mathbf{G}_{Q,d}(\overline{k})$.

Proof. One has $\beta(1_{\mathrm{Gal}_k}) = 1_{\Delta(\overline{k})}$ and $\beta_{\tau_1\tau_2} = \beta_{\tau_1}\tau_1(\beta_{\tau_2})$, so β is a normalised 1-cocycle. If $g' \in \mathbf{G}_{Q,d}(\overline{k})$ also satisfies $g' \cdot M_2 = M_1$, then it follows that $a := g'g^{-1} \in \Delta(\overline{k})$ and, for all $\tau \in \mathrm{Gal}_k$, we have $g'\tau(g')^{-1} = ag\tau(g^{-1})\tau(a^{-1})$; thus the cocyle β' defined using g' instead of g is cohomologous to β. Similarly, if we replace for instance M_1 by $M_1' = u \cdot M_1$ where $u \in \mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k}$, then $g\tau(g^{-1})$ is replaced by $ug\tau((ug)^{-1}) = ug\tau(g^{-1})\tau(u^{-1})$, which yields the same cohomology class as β. And if we replace M_2 by $M_2' = u \cdot M_2$ where $u \in \mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k}$, then $g\tau(g^{-1})$ is replaced by $gu^{-1}\tau((gu^{-1})^{-1}) = g\tau(g^{-1})$, which actually yields the same cocycle β as before. Finally, since by definition $\beta(\tau) = g\tau(g^{-1})$ with $g \in \mathbf{G}_{Q,d}(\overline{k})$, one has that the $\Delta(\overline{k})$-valued 1-cocycle β splits over $\mathbf{G}_{Q,d}(\overline{k})$, i.e. β belongs to the kernel of the pointed map $H^1(\mathrm{Gal}_k; \Delta(\overline{k})) \longrightarrow H^1(\mathrm{Gal}_k, \mathbf{G}_{Q,d}(\overline{k}))$. $\square$

Proof of Proposition 3.3. Let $[M_1] := \mathbf{G}_{Q,d}(k) \cdot M_1 \in \mathrm{Rep}_{Q,d}^{\theta-gs}(k)/\mathbf{G}_{Q,d}(k)$. By Lemma 3.5, there is a map

$$f_{\mathrm{Gal}_k}^{-1}(f_{\mathrm{Gal}_k}([M_1])) \longrightarrow \ker\left(H^1(\mathrm{Gal}_k; \Delta(\overline{k})) \longrightarrow H^1(\mathrm{Gal}_k, \mathbf{G}_{Q,d}(\overline{k}))\right).$$

This map is surjective, as if we have a 1-cocycle $\gamma(\tau) = g\tau(g^{-1}) \in \Delta(\overline{k})$ that splits over $\mathbf{G}_{Q,d}(\overline{k})$, then $\tau(g^{-1} \cdot M_1) = g^{-1} \cdot M_1$, since $\Delta(\overline{k})$ acts trivially on M_1, so the cocycle β defined using M_1 and $M_2 := g^{-1} \cdot M_1$ as above is equal to γ. To prove that the above map is injective, suppose that the $\Delta(\overline{k})$-valued 1-cocycle β associated to

M_1 and $M_2 := g^{-1} \cdot M_1$ splits over $\Delta(\overline{k})$ (i.e. that there exists $a \in \Delta(\overline{k})$ such that $g\tau(g^{-1}) = a\tau(a^{-1})$ for all $\tau \in \mathrm{Gal}_k$). Then, on the one hand, $a^{-1}g \in \mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k}$, as $\tau(a^{-1}g) = a^{-1}g$ for all $\tau \in \mathrm{Gal}_k$, and, on the other hand,

$$(a^{-1}g)^{-1} \cdot M_1 = g^{-1} \cdot (a^{-1} \cdot M_1) = g^{-1} \cdot M_1 = M_2,$$

as $\Delta(\overline{k})$ acts trivially on $\mathrm{Rep}_{Q,d}(\overline{k})$. Therefore, $\mathbf{G}_{Q,d}(k) \cdot M_1 = \mathbf{G}_{Q,d}(k) \cdot M_2$. □

We now turn to the study of the image of the natural map

$$f_{\mathrm{Gal}_k} : F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k).$$

To that end, we introduce a map $\mathcal{T}$ called the type map, from $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k}$ to the Brauer group of k, denoted by $\mathrm{Br}(k)$:

$$\mathcal{T} : \mathcal{M}_{Q,d}^{\theta-gs}(k) \longrightarrow H^2(\mathrm{Gal}_k; \overline{k}^{\times}) \cong \mathrm{Br}(k). \tag{3.6}$$

This map is defined as follows. Consider an orbit

$$(\mathbf{G}_{Q,d}(\overline{k}) \cdot M) \in \mathcal{M}_{Q,d}^{\theta-gs}(k) = (\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})/\mathbf{G}_{Q,d}(\overline{k}))^{\mathrm{Gal}_k},$$

of which a representative M has been chosen. As this orbit is preserved by the Gal_k-action, we have that, for all $\tau \in \mathrm{Gal}_k$, there is an element $u_\tau \in \mathbf{G}_{Q,d}(\overline{k})$ such that $u_\tau \cdot \tau(M) = M$. Note that for $\tau = 1_{\mathrm{Gal}_k}$, we can simply take $u_\tau = 1_{\mathbf{G}_{Q,d}(\overline{k})}$, which we will. Since $(\tau_1\tau_2)(M) = \tau_1(\tau_2(M))$, it follows from the compatibility relation (3.3) that,

$$u_{\tau_1\tau_2}^{-1} \cdot M = \tau_1(u_{\tau_2}^{-1} \cdot M) = \tau_1(u_{\tau_2}^{-1}) \cdot \tau_1(M) = \tau(u_{\tau_2}^{-1})u_{\tau_1}^{-1} \cdot M.$$

Therefore, for all $(\tau_1, \tau_2) \in \mathrm{Gal}_k \times \mathrm{Gal}_k$, the element $c_u(\tau_1, \tau_2) := u_{\tau_1}\tau_1(u_{\tau_2})u_{\tau_1\tau_2}$ (which depends on the choice of the representative M and the family $u := (u_\tau)_{\tau\in\mathrm{Gal}_k}$ satisfying, for all $\tau \in \mathrm{Gal}_k$, $u_\tau \cdot \tau(M) = M$) lies in the stabiliser of M in $\mathbf{G}_{Q,d}(\overline{k})$, which is $\Delta(\overline{k})$ since M is assumed to be χ_θ-stable.

Proposition 3.6. *The above map*

$$\begin{array}{rccc} c_u : & \mathrm{Gal}_k \times \mathrm{Gal}_k & \longrightarrow & \Delta(\overline{k}) \\ & (\tau_1, \tau_2) & \longmapsto & u_{\tau_1}\tau_1(u_{\tau_2})u_{\tau_1\tau_2}^{-1} \end{array}$$

is a normalised $\Delta(\overline{k})$-valued 2-cocycle whose cohomology class only depends on the $\mathbf{G}_{Q,d}(\overline{k})$-orbit of M, thus this defines a map

$$\mathcal{T} : \mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k} \longrightarrow H^2(\mathrm{Gal}_k; \Delta(\overline{k})) \simeq \mathrm{Br}(k)$$

that we shall call the type map.

Proof. It is straightforward to check the cocycle relation

$$c(\tau_1, \tau_2)c(\tau_1\tau_2, \tau_3) = \tau_1(c(\tau_2, \tau_3))c(\tau_1, \tau_2\tau_3)$$

for all τ_1, τ_2, τ_3 in Gal_k. If we choose a different family $u' := (u'_\tau)_{\tau\in\mathrm{Gal}_k}$ such that $u'_\tau \cdot \tau(M) = M$ for all $\tau \in \mathrm{Gal}_k$, then $(u'_\tau)^{-1} \cdot M = u_\tau \cdot M$, thus $a_\tau := u'_\tau u_\tau^{-1} \in \Delta(\overline{k})$ and it is straightforward to check, using that $\Delta(\overline{k})$ is a central subgroup of $\mathbf{G}_{Q,d}(\overline{k})$, that

$$u'_{\tau_1}\tau_1(u'_{\tau_2})\big(u'_{\tau_1\tau_2}\big)^{-1} = \big(a_{\tau_1}\tau_1(a_{\tau_2})a_{\tau_1\tau_2}^{-1}\big)\ \big(u_{\tau_1}\tau_1(u_{\tau_2})u_{\tau_1\tau_2}^{-1}\big).$$

Therefore, the associated cocycles c_u and $c_{u'}$ are cohomologous. If we now replace M with $M' = g \cdot M$ for $g \in \mathbf{G}_{Q,d}(\overline{k})$, then

$$\tau(M') = \tau(g) \cdot \tau(M) = \tau(g)u_\tau^{-1}g^{-1} \cdot M'$$

and, if we set $u'_\tau := g u_\tau \tau(g^{-1})$, we have

$$c_{u'}(\tau_1,\tau_2) = g c_u(\tau_1,\tau_2) g^{-1} = c_u(\tau_1,\tau_2),$$

where the last equality follows again from the fact that $\Delta(\overline{k})$ is central in $\mathbf{G}_{Q,d}(\overline{k})$. In particular, the two representatives M and M' give rise, for an appropriate choice of the families u and u', to the same cocycle, and thus they induce the same cohomology class $[c_u] = [c_{u'}]$. □

If k is a finite field, $\mathrm{Br}(k) = \{1\}$. Other useful examples of target spaces for the type map are $\mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Br}(\mathbb{Q}_p) \simeq \mathbb{Q}/\mathbb{Z}$ for all prime p.

Remark 3.7 (Intrinsic definition of the type map)**.** The presentation of $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ as the orbit space $\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})/\mathbf{G}_{Q,d}(\overline{k})$ is particularly well-suited for defining the type map, as the stabiliser in $\mathbf{G}_{Q,d}(\overline{k})$ of a point in $\mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})$ is isomorphic to the automorphism group of the associated representation of Q. We can intrinsically define the type map, without using this orbit space presentation, as follows. A point in $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k})$ corresponds to an isomorphism class of a θ-geometrically stable $\overline{k}$-representation W, and this point is fixed by Gal_k-action if, for all $\tau \in \mathrm{Gal}_k$, there is an isomorphism $u_\tau : W \longrightarrow W^\tau$. The relation $W^{\tau_1\tau_2} = (W^{\tau_1})^{\tau_2}$ then implies that $\tilde{c}_u(\tau_1,\tau_2) := u_{\tau_1\tau_2}^{-1} u_{\tau_1}^{\tau_2} u_{\tau_2}$ is an automorphism of W. Once $\mathrm{Aut}(W)$ is identified with $\overline{k}^\times$, this defines a $\overline{k}^\times$-valued 2-cocycle $\tilde{c}_u$, whose cohomology class is independent of the choice of the isomorphisms $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ and the identification $\mathrm{Aut}(W) \simeq \overline{k}^\times$.

We now use the type map to analyse which k-points of the moduli scheme $\mathcal{M}_{Q,d}^{\theta-gs}$ actually correspond to k-representations of Q.

Theorem 3.8. *The natural map $F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k)$ induces a bijection*

$$F_{Q,d}^{\theta-gs}(k) \xrightarrow{\simeq} \mathcal{T}^{-1}([1]) \subset \mathcal{M}_{Q,d}^{\theta-gs}(k)$$

from the set of isomorphism classes of θ-geometrically stable d-dimensional k-representations of Q onto the fibre of the type map $\mathcal{T} : \mathcal{M}_{Q,d}^{\theta-gs}(k) \longrightarrow \mathrm{Br}(k)$ over the trivial element of the Brauer group of k.

Proof. Identify this map with f_{Gal_k}; then it is injective by Corollary 3.4. If $\mathbf{G}_{Q,d}(\overline{k})\cdot M$ lies in $\mathrm{Im}\, f_{\mathrm{Gal}_k}$, we can choose a representative $M \in \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k}$, so the relation $u_\tau \cdot \tau(M) = M$ is trivially satisfied if we set $u_\tau = 1_{\mathrm{Gal}_k}$ for all $\tau \in \mathrm{Gal}_k$. But then $c_u(\tau_1,\tau_2) \equiv 1_{\Delta(\overline{k})}$ so, by definition of the type map, $\mathcal{T}(\mathbf{G}_{Q,d}(\overline{k})\cdot M) = [c_u] = [1]$, which proves that $\mathrm{Im}\, f_{\mathrm{Gal}_k} \subset \mathcal{T}^{-1}([1])$. Conversely, take $M \in \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})$ with $\mathbf{G}_{Q,d}(\overline{k})\cdot M \in \mathcal{T}^{-1}([1])$. By definition of the type map, this means that there exists a family $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ of elements of $\mathbf{G}_{Q,d}(\overline{k})$ such that $u_{1_{\mathrm{Gal}_k}} = 1_{\mathbf{G}_{Q,d}(\overline{k})}$, $u_\tau \cdot \tau(M) = M$ for all $\tau \in \mathrm{Gal}_k$ and $c_u(\tau_1,\tau_2) := u_{\tau_1}\tau_1(u_{\tau_2})u_{\tau_1\tau_2}^{-1} \in \Delta(\overline{k})$ for all $(\tau_1,\tau_2) \in \mathrm{Gal}_k \times \mathrm{Gal}_k$, and $[c_u] = [1]$, as $\mathcal{T}(\mathbf{G}_{Q,d}(\overline{k})\cdot M) = [c_u]$ by construction of $\mathcal{T}$. By suitably modifying the family $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ if necessary, we can thus assume that $u_{\tau_1}\tau_1(u_{\tau_2}) = u_{\tau_1\tau_2}$, which means that $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ is a $\mathbf{G}_{Q,d}(\overline{k})$-valued 1-cocycle for Gal_k. Since $\mathbf{G}_{Q,d}(\overline{k}) = \prod_{v\in V} \mathbf{GL}_{d_v}(\overline{k})$, we have

$$H^1(\mathrm{Gal}_k; \mathbf{G}_{Q,d}(\overline{k})) \simeq \prod_{v\in V} H^1(\mathrm{Gal}_k; \mathbf{GL}_{d_v}(\overline{k}))$$

so, by a well-known generalisation of Hilbert's 90 (for instance, see [32, Proposition X.1.3 p.151]), $H^1(\mathrm{Gal}_k; \mathbf{G}_{Q,d}(\overline{k})) = 1$. Therefore, there exists $g \in \mathbf{G}_{Q,d}(\overline{k})$ such that $u_\tau = g\tau(g^{-1})$ for all $\tau \in \mathrm{Gal}_k$. In particular, the relation $u_\tau \cdot \tau(M) = M$ implies that $\tau(g^{-1} \cdot M) = g^{-1} \cdot M$, i.e. $(g^{-1} \cdot M) \in \mathrm{Rep}_{Q,d}(\overline{k})^{\mathrm{Gal}_k}$, which shows that $\mathcal{T}^{-1}([1]) \subset \mathrm{Im}\, f_{\mathrm{Gal}_k}$. □

Example 3.9. If k is a finite field (so, in particular, k is perfect and $\mathrm{Br}(k) = 1$), then $F_{Q,d}^{\theta-gs}(k) \simeq \mathcal{M}_{Q,d}^{\theta-gs}(k)$: the set of isomorphism classes of θ-geometrically stable d-dimensional k-representations of Q is the set of k-points of a k-variety $\mathcal{M}_{Q,d}^{\theta-gs}$.

3.3. **Rational points that do not come from rational representations.** When the Brauer group of k is non-trivial, the type map $\mathcal{T} : \mathcal{M}_{Q,d}^{\theta-gs}(k) \longrightarrow \mathrm{Br}(k)$ can have non-empty fibres other than $\mathcal{T}^{-1}([1])$ (see Example 3.13). In particular, by Theorem 3.8, the natural map $F_{Q,d}^{\theta-gs}(k) \longrightarrow \mathcal{M}_{Q,d}^{\theta-gs}(k)$ is injective but not surjective in that case. The goal of the present section is to show that the fibres of the type map above non-trivial elements of the Brauer group of k still admit a modular interpretation.

If $[c] \in H^2(\mathrm{Gal}_k; \overline{k}^\times)$ lies in the image of the type map, then by definition there exists a representation $M \in \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})$ and a family $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ such that $u_{1_{\mathrm{Gal}_k}} = 1_{\mathbf{G}_{Q,d}(\overline{k})}$ and $u_\tau \cdot \Phi_\tau(M) = M$ for all $\tau \in \mathrm{Gal}_k$. Moreover, the given 2-cocycle c is cohomologous to the 2-cocycle $c_u : (\tau_1, \tau_2) \longmapsto u_{\tau_1}\Psi_{\tau_1}(u_{\tau_2})u_{\tau_1\tau_2}^{-1}$. In order to analyse such families $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ in detail, we introduce the following terminology, reflecting the fact that these families will later be used to modify the Gal_k-action on $\mathrm{Rep}_{Q,d}(\overline{k})$ and $\mathbf{G}_{Q,d}(\overline{k})$.

Definition 3.10 (Modifying family)**.** A modifying family $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ is a tuple, indexed by Gal_k, of elements $u_\tau \in \mathbf{G}_{Q,d}(\overline{k})$ satisfying:

(1) $u_{1_{\mathrm{Gal}_k}} = 1_{\mathbf{G}_{Q,d}(\overline{k})}$;
(2) For all $(\tau_1, \tau_2) \in \mathrm{Gal}_k \times \mathrm{Gal}_k$, the element $c_u(\tau_1, \tau_2) := u_{\tau_1}\Psi_{\tau_1}(u_{\tau_2})u_{\tau_1\tau_2}^{-1}$ lies in the subgroup $\Delta(\overline{k}) \subset \mathbf{G}_{Q,d}(\overline{k})$.

In particular, if $u = (u_\tau)_{\tau\in\mathrm{Gal}_k}$ is a modifying family, then the induced map

$$c_u : \mathrm{Gal}_k \times \mathrm{Gal}_k \longrightarrow \Delta(\overline{k})$$

is a normalised $\Delta(\overline{k})$-valued 2-cocycle. We now show that a modifying family can indeed be used to define new Gal_k-actions on $\mathrm{Rep}_{Q,d}(\overline{k})$ and $\mathbf{G}_{Q,d}(\overline{k})$.

Proposition 3.11. *Let $u = (u_\tau)_{\tau\in\mathrm{Gal}_k}$ be a modifying family in the sense of Definition 3.10. Then we can define modified Gal_k-actions*

$$\begin{array}{cccc} \Phi^u : & \mathrm{Gal}_k \times \mathrm{Rep}_{Q,d}(\overline{k}) & \longrightarrow & \mathrm{Rep}_{Q,d}(\overline{k}) \\ & (\tau, M) & \longmapsto & u_\tau \cdot \Phi_\tau(M) \end{array}$$

and

$$\begin{array}{cccc} \Psi^u : & \mathrm{Gal}_k \times \mathbf{G}_{Q,d}(\overline{k}) & \longrightarrow & \mathbf{G}_{Q,d}(\overline{k}) \\ & (\tau, g) & \longmapsto & u_\tau\Psi_\tau(g)u_\tau^{-1} \end{array}$$

which are compatible in the sense of (3.3) *and such that the induced* Gal_k*-actions on* $\mathcal{M}_{Q,d}^{\theta-ss}(\overline{k}) \simeq \mathrm{Rep}_{Q,d}(\overline{k})//_{\chi_\theta}\mathbf{G}_{Q,d}(\overline{k})$ *and* $\mathcal{M}_{Q,d}^{\theta-gs}(\overline{k}) \simeq \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})/\mathbf{G}_{Q,d}(\overline{k})$ *coincide with the previous ones, constructed in* (3.4).

Proof. The proof is a simple verification, using the fact that $\Delta(\overline{k})$ acts trivially on $\mathrm{Rep}_{Q,d}(\overline{k})$ and is central in $\mathbf{G}_{Q,d}(\overline{k})$, then proceeding as in Propositions 3.1 and 3.2 to show that the modified Gal_k-action is compatible with semistability and stability of $\overline{k}$-representations. □

Let us denote by $\mathrm{Rep}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$ the fixed-point set of Φ^u in $\mathrm{Rep}_{Q,d}(\overline{k})$ and by $\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$ the fixed subgroup of $\mathbf{G}_{Q,d}(\overline{k})$ under Ψ^u. Proposition 3.11 then immediately implies that $\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$ acts on $\mathrm{Rep}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$ and that the map $f_{\mathrm{Gal}_k,u}$ taking the $\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}$-orbit of a θ-geometrically stable representation $M \in \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k,u}$ to its $\mathbf{G}_{Q,d}(\overline{k})$-orbit in $\mathcal{M}_{Q,d}^{\theta-s}(\overline{k})$ lands in $\mathcal{T}^{-1}([c_u])$, since one has $u_\tau \cdot \tau(M) = M$ for such a representation. We then have the following generalisation of Theorem 3.8.

Theorem 3.12. *Let $(u_\tau)_{\tau\in\mathrm{Gal}_k}$ be a modifying family in the sense of Definition 3.10 and let $c_u : \mathrm{Gal}_k \times \mathrm{Gal}_k \longrightarrow \Delta(\overline{k}) \simeq \overline{k}^\times$ be the associated 2-cocycle. Then the map*

$$\begin{array}{rccc} f_{\mathrm{Gal}_k,u} : & \mathrm{Rep}_{Q,d}^{\theta-gs}(\overline{k})^{\mathrm{Gal}_k,u}/\mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u} & \longrightarrow & \mathcal{T}^{-1}([c_u]) \\ & \mathbf{G}_{Q,d}(\overline{k})^{\mathrm{Gal}_k,u}\cdot M & \longmapsto & \mathbf{G}_{Q,d}(\overline{k})\cdot M \end{array}$$

is bijective.

Proof. As $\Delta(\overline{k})$ is central in $\mathbf{G}_{Q,d}(\overline{k})$, the action induced by Ψ^u on $\Delta(\overline{k})$ coincides with the one induced by Ψ, so the injectivity of $f_{\mathrm{Gal}_k,u}$ can be proved as in Corollary 3.4. The proof of surjectivity is then exactly the same as in Theorem 3.8. The only thing to check is that $H^1_u(\mathrm{Gal}_k; \mathbf{G}_{Q,d}(\overline{k})) = 1$, where the subscript u means that Gal_k now acts on $\mathbf{G}_{Q,d}(\overline{k})$ via the action Ψ^u; this follows from the proof of [32, Proposition X.1.3 p.151] once one observes that, if one sets $\Psi^u_\tau(x) := u_\tau \tau(x)$ for all $x \in \overline{k}^{d_v}$, then one still has, for all $A \in \mathbf{GL}_{d_v}(\overline{k})$ and all $x \in \overline{k}^{d_v}$, $\Psi^u_\tau(Ax) = \Psi^u_\tau(A)\Psi^u_\tau(x)$. After that, the proof is the same as in *loc. cit.*. □

Theorem 3.12 provides the desired modular interpretation of $\mathcal{T}^{-1}([c_u])$: elements of that fibre are isomorphism classes of θ-geometrically stable, (Gal_k, u)-invariant, d-dimensional $\overline{k}$-representations of Q, as illustrated by the following example.

Example 3.13. Let $k = \mathbb{R}$ and let $[c] = -1 \in \mathrm{Br}(\mathbb{R}) \simeq \{\pm 1\}$. Then a modifying family corresponds to an element $u \in \mathbf{G}_{Q,d}(\mathbb{C}) = \prod_{v\in V} \mathbf{GL}_{d_v}(\mathbb{C})$ such that, for all $v \in V$, $u_v\overline{u_v} = -I_{d_v}$, implying that $|\det u_v|^2 = (-1)^{d_v}$, which can only happen if d_v is even for all $v \in V$. We then have a quaternionic structure on each $\mathbb{C}^{d_v}$, given by $x \longmapsto u_v\overline{x}$ and a modified $\mathrm{Gal}_\mathbb{R}$-action on $\mathrm{Rep}_{Q,d}(\mathbb{C})$, given by $(M_a)_{a\in A} \longmapsto u_{h(a)}\overline{M_a}u_{t(a)}^{-1}$. The fixed points of this involution are those $(M_a)_{a\in A}$ satisfying $u_{h(a)}\overline{M_a}u_{t(a)}^{-1} = M_a$, i.e. those $\mathbb{C}$-linear maps $M_a : W_{t(a)} \longrightarrow W_{h(a)}$ that commute with the quaternionic structures defined earlier. The subgroup of $\mathbf{G}_{Q,d}(\mathbb{C}) = \prod_{v\in V} \mathbf{GL}_{d_v}(\mathbb{C})$ consisting, for each $v \in V$, of automorphisms of the quaternionic structure of $\mathbb{C}^{d_v}$ is the real Lie group $\mathbf{G}_{Q,d}(\mathbb{C})^{(\mathrm{Gal}_\mathbb{R},u)} = \prod_{v\in V} \mathbf{U}^*(d_v)$. The fibre $\mathcal{T}^{-1}(-1)$ of the type map is in bijection with $\mathrm{Rep}_{Q,d}^{\theta-gs}(\mathbb{C})^{(\mathrm{Gal}_\mathbb{R},u)}/\mathbf{G}_{Q,d}(\mathbb{C})^{(\mathrm{Gal}_\mathbb{R},u)}$, i.e. isomorphism classes of θ-geometrically stable representations of the quiver Q by homomorphisms between quaternionic vector spaces.

Note that, in the context of (Gal_k, u)-invariant $\overline{k}$-representations of Q, semistability is defined with respect to (Gal_k, u)-invariant $\overline{k}$-subrepresentations only. However, analogously to Proposition 2.4, this is in fact equivalent to semistability with respect to all subrepresentations. The same holds for geometric stability, by definition. We have thus obtained a decomposition of the set of k-points of $\mathcal{M}_{Q,d}^{\theta-gs}$ as a disjoint union of moduli spaces, completing the proof of Theorem 1.1.

4. Automorphisms of quivers

Definition 4.1. A covariant (resp. contravariant) automorphism of $Q = (V, A, h, t)$ is a pair of bijections $(\sigma_V : V \longrightarrow V, \sigma_A : A \longrightarrow A)$ such that

$$\begin{array}{llll} t \circ \sigma_A = \sigma_V \circ t & \text{and} & h \circ \sigma_A = \sigma_V \circ h & \text{if } \sigma \text{ is covariant,} \\ t \circ \sigma_A = \sigma_V \circ h & \text{and} & h \circ \sigma_A = \sigma_V \circ t & \text{if } \sigma \text{ is contrariant.} \end{array} \tag{4.1}$$

Henceforth, to simplify notation, we denote σ_V and σ_A both by σ. We let $\mathrm{Aut}^+(Q)$ (resp. $\mathrm{Aut}^-(Q)$) denote the set of covariant (resp. contravariant) automorphisms of Q and write $\mathrm{Aut}(Q) := \mathrm{Aut}^+(Q) \sqcup \mathrm{Aut}^-(Q)$.

Note that, for any field k, a covariant (resp. contravariant) automorphism σ of Q determines a graded algebra automorphism (resp. anti-automorphism) of the path algebra kQ. We refer to the covariant automorphism of Q given by $\sigma_V = \mathrm{Id}_V, \sigma_A = \mathrm{Id}_A$ as the trivial automorphism of Q. As the composition of covariant automorphisms is covariant, $\mathrm{Aut}^+(Q)$ is a subgroup of $\mathrm{Aut}(Q)$. There is a group homomorphism $\mathrm{sign} : \mathrm{Aut}(Q) \longrightarrow \{\pm 1\}$ sending σ to -1 if and only if σ is contravariant. Evidently, $\ker \mathrm{sign} = \mathrm{Aut}^+(Q)$.

For a subgroup $\Sigma \subset \mathrm{Aut}(Q)$ of quiver automorphisms, we want to study induced actions of Σ on the moduli space $\mathcal{M}_{Q,d}^{\theta-ss}$ of θ-semistable d-dimensional k-representations of Q (provided d and θ are Σ-compatible in the sense of Definitions 4.2 and 4.35). For a subgroup $\Sigma \subset \mathrm{Aut}(Q)$, either $\Sigma \subset \mathrm{Aut}^+(Q)$ or Σ is an extension

$$1 \longrightarrow \Sigma^+ \longrightarrow \Sigma \longrightarrow \{\pm 1\} \longrightarrow 1,$$

where $\Sigma^+ \subset \mathrm{Aut}^+(Q)$ is a subgroup of covariant automorphisms. Therefore, one should start by studying the actions by subgroups of covariant automorphisms and actions by contravariant involutions. Since the latter is studied in [35], we restrict our attention to subgroups $\Sigma \subset \mathrm{Aut}^+(Q)$ of covariant automorphisms until §4.5

4.1. The induced action on the moduli space.

Let $\Sigma \subset \mathrm{Aut}^+(Q)$ be a subgroup of covariant automorphisms and let k be a field. In this section, we will construct algebraic Σ-actions on moduli spaces $\mathcal{M}_{Q,d}^{\theta-ss}$ of θ-semistable d-dimensional k-representations of Q, when d and θ are Σ-compatible in the following sense.

Definition 4.2. Let us denote $\sigma(d) := (d_{\sigma(v)})_{v\in V}$ and $\sigma(\theta) := (\theta_{\sigma(v)})_{v\in V}$.

1. A dimension vector $d = (d_v)_{v\in V}$ is Σ-compatible if $\sigma(d) = d$ for all $\sigma \in \Sigma$.
2. A stability parameter $\theta = (\theta_v)_{v\in V}$ is Σ-compatible if $\sigma(\theta) = \theta$ for all $\sigma \in \Sigma$.

Throughout the rest of Section 4, we assume that d and θ are Σ-compatible. To construct the Σ-action on $\mathcal{M}_{Q,d}^{\theta-ss}$, we use the GIT construction of this moduli space (Corollary 2.12). As d is Σ-compatible, we have algebraic actions

$$\Phi : \Sigma \times \mathrm{Rep}_{Q,d} \longrightarrow \mathrm{Rep}_{Q,d} \quad \text{and} \quad \Psi : \Sigma \times \mathbf{G}_{Q,d} \longrightarrow \mathbf{G}_{Q,d}$$

given by, for $\sigma \in \Sigma$ and for $M := (M_a)_{a\in A} \in \mathrm{Rep}_{Q,d}$ and $g := (g_v)_{v\in V} \in \mathbf{G}_{Q,d}$,

$$\Phi_\sigma(M) := \sigma(M) := (M_{\sigma(a)})_{a\in A} \quad \text{and} \quad \Psi_\sigma(g) := \sigma(g) := (g_{\sigma(v)})_{v\in V}.$$

These actions are compatible with each other in the sense that

$$\Phi_\sigma(g \cdot M) = \Psi_\sigma(g) \cdot \Phi_\sigma(M). \tag{4.2}$$

Proposition 4.3. *For Σ-compatible d and θ, the following statements hold.*

(1) *The Σ-action on $\mathrm{Rep}_{Q,d}$ preserves the GIT (semi)stable sets $\mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s}$;*
(2) *There is an induced algebraic Σ-action on the moduli spaces $\mathcal{M}_{Q,d}^{\theta-(gs)s}$.*

Proof. For (1), it suffices to check that, for all $\sigma \in \Sigma$, the image of each $M \in \mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s}(\bar{k})$ under Φ_σ lies in $\mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s}(\bar{k})$; this follows from the same techniques as in Propositions 3.1 and 3.2, noting that the character χ_θ is Σ-invariant.

For (2), recall that $\pi : \mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s} \longrightarrow \mathcal{M}_{Q,d}^{\theta-(gs)s}$ is a categorical quotient of the $\mathbf{G}_{Q,d}$-action. To define the induced action $\Phi' : \Sigma \times \mathcal{M}_{Q,d}^{\theta-(gs)s} \longrightarrow \mathcal{M}_{Q,d}^{\theta-(gs)s}$, we observe that, for each $\sigma \in \Sigma$, the morphism $\pi \circ \Phi_\sigma$ is $\mathbf{G}_{Q,d}$-invariant (because of (4.2)), so there is a unique morphism Φ'_σ making the following diagram commute

$$\begin{array}{ccc} \mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s} & \xrightarrow{\Phi_\sigma} & \mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{M}_{Q,d}^{\theta-(gs)s} & \xrightarrow{\Phi'_\sigma} & \mathcal{M}_{Q,d}^{\theta-(gs)s}, \end{array} \tag{4.3}$$

given by the universal property of the categorical quotient π. □

4.2. **Morphisms to the fixed-point set of the action.** For $\Sigma \subset \mathrm{Aut}^+(Q)$, we constructed an algebraic Σ-action on $\mathcal{M}_{Q,d}^{\theta-ss}$ (provided d and θ are Σ-compatible). The fixed locus $(\mathcal{M}_{Q,d}^{\theta-ss})^\Sigma$ is a closed k-subscheme of $\mathcal{M}_{Q,d}^{\theta-ss}$ and in this section we construct morphisms from related moduli spaces to this fixed locus.

Definition 4.4 (Quotient quiver)**.** Given $\Sigma \subset \mathrm{Aut}^+(Q)$, we define the quotient quiver $Q/\Sigma := (V/\Sigma, A/\Sigma, \tilde{h}, \tilde{t})$, where V/Σ and A/Σ denote the set of Σ-orbits in V and A respectively, and the head and tail maps $\tilde{h}, \tilde{t} : A/\Sigma \longrightarrow V/\Sigma$ are given by $\tilde{h}(\Sigma \cdot a) = \Sigma \cdot h(a)$ and $\tilde{t}(\Sigma \cdot a) = \Sigma \cdot t(a)$, which are well-defined by (4.1).

A Σ-compatible dimension vector d and stability parameter θ for Q determine a dimension vector $\tilde{d}$ and stability parameter $\tilde{\theta}$ for Q/Σ, where $\tilde{d}_{\Sigma \cdot v} := d_v$ and $\tilde{\theta}_{\Sigma\cdot v} := |\Sigma \cdot v|\theta_v$. Let $\mathbf{G}_{Q,d}^\Sigma$ and $\mathrm{Rep}_{Q,d}^\Sigma$ denote the fixed loci for the actions of Σ on $\mathbf{G}_{Q,d}$ and $\mathrm{Rep}_{Q,d}$ given by Ψ and Φ respectively; then the action of $\mathbf{G}_{Q,d}^\Sigma$ on $\mathrm{Rep}_{Q,d}$ preserves $\mathrm{Rep}_{Q,d}^\Sigma$ by (4.2). By the following lemma, $\mathbf{G}_{Q,d}^\Sigma$ is reductive and so we can consider GIT quotients for the $\mathbf{G}_{Q,d}^\Sigma$-action on $\mathrm{Rep}_{Q,d}^\Sigma$.

Proposition 4.5. *There are isomorphisms*

$$\alpha : \mathbf{G}_{Q/\Sigma,\tilde{d}} \xrightarrow{\simeq} \mathbf{G}_{Q,d}^\Sigma \quad \textit{and} \quad \beta : \mathrm{Rep}_{Q/\Sigma,\tilde{d}} \xrightarrow{\simeq} \mathrm{Rep}_{Q,d}^\Sigma$$

such that, if the two groups are identified through α, then β is equivariant with respect to the $\mathbf{G}_{Q/\Sigma,\tilde{d}}$-action on $\mathrm{Rep}_{Q/\Sigma,\tilde{d}}$ and the $\mathbf{G}_{Q,d}^\Sigma$-action on $\mathrm{Rep}_{Q,d}^\Sigma$.

Proof. The isomorphisms are defined by

$$\alpha((g_{\Sigma\cdot v})_{\Sigma\cdot v \in V/\Sigma}) = (g_{\Sigma\cdot v})_{v\in V} \quad \text{and} \quad \beta((M_{\Sigma\cdot a})_{\Sigma\cdot a\in A/\Sigma}) = (M_{\Sigma\cdot a})_{a\in A},$$

which are compatible with the actions by construction. □

Corollary 4.6. *There are isomorphisms*

$$\mathrm{Rep}_{Q,d}^{\Sigma} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma} \cong \mathrm{Rep}_{Q/\Sigma,\tilde{d}} /\!/_{\chi_{\tilde{\theta}}} \mathbf{G}_{Q/\Sigma,\tilde{d}}$$

and $(\mathrm{Rep}_{Q,d}^{\Sigma})^{\chi_\theta - s}/\mathbf{G}_{Q,d}^{\Sigma} \cong \mathrm{Rep}_{Q/\Sigma,\tilde{d}}^{\chi_{\tilde{\theta}}-s}/\mathbf{G}_{Q/\Sigma,\tilde{d}}$ *where, on the left hand side of these morphisms,* χ_θ *denotes the restriction to* $\mathbf{G}_{Q,d}^{\Sigma}$ *of the character* χ_θ *of* $\mathbf{G}_{Q,d}$*. Consequently, the GIT quotient* $\mathrm{Rep}_{Q,d}^{\Sigma} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma}$ *(resp.* $(\mathrm{Rep}_{Q,d}^{\Sigma})^{\chi_\theta - s}/\mathbf{G}_{Q,d}^{\Sigma}$*) can be interpreted as the moduli space* $\mathcal{M}_{Q/\Sigma,\tilde{d}}^{\tilde{\theta}-(gs)s}$ *of* $\tilde{\theta}$*-semistable (resp.* $\tilde{\theta}$*-geometrically stable)* $\tilde{d}$*-dimensional representations of* Q/Σ*.*

Proof. The isomorphisms are a consequence of Proposition 4.5, the universal property of the categorical quotient and the observation that $\chi_{\tilde{\theta}} = \chi_\theta \circ \alpha$ (since, by definition, $\tilde{\theta}_{\Sigma\cdot v} = |\Sigma \cdot v|\theta_v$). The last assertion then follows from Corollary 2.12. □

In Corollary 4.34, we will provide another modular interpretation of the GIT quotient $\mathrm{Rep}_{Q,d}^{\Sigma} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma}$.

We phrase the following lemma in a slightly more general setting and note, for $\Sigma \subset \mathrm{Aut}^+(Q)$, that Σ-compatibility of θ is equivalent to χ_θ being Σ-invariant.

Lemma 4.7. *Let* Σ *be a finite group acting algebraically on* $\mathrm{Rep}_{Q,d}$ *and* $\mathbf{G}_{Q,d}$ *in a compatible manner and fix a stability parameter* θ *such that* χ_θ *is* Σ*-invariant. Then, for a* Σ*-fixed geometric point* $M : \mathrm{Spec}\,\Omega \longrightarrow \mathrm{Rep}_{Q,d}$*, the following statements are equivalent.*

1. *M is GIT semistable for the* $\mathbf{G}_{Q,d}$*-action on* $\mathrm{Rep}_{Q,d}$ *with respect to the character* $\chi_\theta : \mathbf{G}_{Q,d} \longrightarrow \mathbb{G}_m$*.*
2. *M is GIT semistable for the* $\mathbf{G}_{Q,d}^{\Sigma}$*-action on* $\mathrm{Rep}_{Q,d}^{\Sigma}$ *with respect to the restricted character* $\chi_\theta : \mathbf{G}_{Q,d}^{\Sigma} \longrightarrow \mathbb{G}_m$*.*

Proof. Evidently, (1) implies (2). For the converse, we proceed by contrapositive and argue along the lines of [17, Theorem 4.2], which is about Galois actions but carries over naturally to our setting. So let us assume that $M \in \mathrm{Rep}_{Q,d}^{\Sigma}(\Omega)$ is not $(\mathbf{G}_{Q,d}, \chi_\theta)$-semistable. Let Λ_M denote the set of 1-PSs λ of $\mathbf{G}_{Q,d,\Omega}$ for which the morphism $\mathbb{G}_{m,\Omega} \longrightarrow \mathrm{Rep}_{Q,d,\Omega}$ given by the λ-action on M extends to $\mathbb{A}^1_\Omega$. By [17, Theorem 3.4], there is a canonical parabolic subgroup $P_M \subset \mathbf{G}_{Q,d,\Omega}$ such that $P_M = P(\lambda)$ for any 1-PS $\lambda \in \Lambda_M$ which minimises a normalised Hilbert–Mumford functional $a_M : \Lambda_M \longrightarrow \mathbb{R}$ given by $a_M(\lambda) =< \chi_\theta, \lambda >/||\lambda||$, where $||-||$ is a length function on the 1-PSs of $\mathbf{G}_{Q,d}$ in the sense of [17, p. 305]. Since M is fixed by Σ, the set of 1-PSs Λ_M is Σ-invariant by the compatibility of the actions of Σ and $\mathbf{G}_{Q,d}$. By summing over Σ (or applying Kempf's construction of length functions to the reductive group $\mathbf{G}_{Q,d} \rtimes \Sigma$), we can assume that $||-||$ is Σ-invariant. Then, as χ_θ is Σ-invariant, it follows that a_M is Σ-invariant analogously to Lemma 4.1 in *loc. cit.* Therefore, $P_M = \sigma(P_M)$ for all $\sigma \in \Sigma$, by the uniqueness of P_M analogously to the proof of Part (b) of Theorem 4.2 in *loc. cit.* Hence, $P_M \subset \mathbf{G}_{Q,d,\Omega}^{\Sigma}$ and so M is not $(\mathbf{G}_{Q,d}^{\Sigma}, \chi_\theta)$-semistable. □

Proposition 4.8. *In* $\mathrm{Rep}_{Q,d}^{\Sigma}$*, there is an equality of* k*-subschemes*

$$(\mathrm{Rep}_{Q,d}^{\Sigma})^{(\mathbf{G}_{Q,d}^{\Sigma},\chi_\theta)-ss} = \mathrm{Rep}_{Q,d}^{\Sigma} \times_{\mathrm{Rep}_{Q,d}} \mathrm{Rep}_{Q,d}^{(\mathbf{G}_{Q,d},\chi_\theta)-ss}.$$

The closed immersion $(\mathrm{Rep}^{\Sigma}_{Q,d})^{(\mathbf{G}^{\Sigma}_{Q,d},\chi_\theta)-ss} \hookrightarrow \mathrm{Rep}^{(\mathbf{G}_{Q,d},\chi_\theta)-ss}_{Q,d}$ *induces a morphism*

$$f_\Sigma : \mathrm{Rep}^{\Sigma}_{Q,d} /\!/_{\chi_\theta} (\mathbf{G}^{\Sigma}_{Q,d}) \longrightarrow \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}$$

whose image is contained in the Σ*-fixed locus.*

Proof. The k-subschemes $(\mathrm{Rep}^{\Sigma}_{Q,d})^{(\mathbf{G}^{\Sigma}_{Q,d},\chi_\theta)-ss}$ and $\mathrm{Rep}^{\Sigma}_{Q,d} \times_{\mathrm{Rep}_{Q,d}} \mathrm{Rep}^{(\mathbf{G}_{Q,d},\chi_\theta)-ss}_{Q,d}$ are open in $\mathrm{Rep}^{\Sigma}_{Q,d}$; therefore, to show that they agree, it suffices to check the equality on $\overline{k}$-points, for which we can use Lemma 4.7. Then the closed immersion

$$(\mathrm{Rep}^{\Sigma}_{Q,d})^{(\mathbf{G}^{\Sigma}_{Q,d},\chi_\theta)-ss} \hookrightarrow \mathrm{Rep}^{(\mathbf{G}_{Q,d},\chi_\theta)-ss}_{Q,d},$$

induces the morphism f_Σ via the universal property of the categorical quotient. It is straightforward to check that the image of f_Σ is contained in the Σ-fixed locus. □

Due to Corollary 4.6, we can interpret f_Σ as a morphism $\mathcal{M}^{\tilde{\theta}-ss}_{Q/\Sigma,\tilde{d}} \longrightarrow \mathcal{M}^{\theta-ss}_{Q,d}$. Let us now study properties of f_Σ (or strictly speaking a restriction f^{rs}_Σ of f_Σ as introduced below) in terms of the group cohomology of Σ.

Definition 4.9 (Regularly stable point)**.** Let Σ be a finite group acting algebraically on $\mathrm{Rep}_{Q,d}$, compatibly with the action of $\mathbf{G}_{Q,d}$. A point $M \in \mathrm{Rep}_{Q,d}$ which is both Σ-fixed and $(\mathbf{G}_{Q,d},\chi_\theta)$-stable is called a (Σ,χ_θ)-regularly stable point of $\mathrm{Rep}_{Q,d}$.

Let

$$(\mathrm{Rep}^{\Sigma}_{Q,d})^{\chi_\theta-rs} := \mathrm{Rep}^{\Sigma}_{Q,d} \times_{\mathrm{Rep}_{Q,d}} \mathrm{Rep}^{(\mathbf{G}_{Q,d},\chi_\theta)-s}_{Q,d}$$

be the $\mathbf{G}^{\Sigma}_{Q,d}$-invariant open subset of $\mathrm{Rep}^{\Sigma}_{Q,d}$ whose points are both Σ-fixed and $(\mathbf{G}_{Q,d},\chi_\theta)$-stable. As $(\mathbf{G}_{Q,d},\chi_\theta)$-stability implies $(\mathbf{G}^{\Sigma}_{Q,d},\chi_\theta)$-stability, we have

$$(\mathrm{Rep}^{\Sigma}_{Q,d})^{\chi_\theta-rs} \subset (\mathrm{Rep}^{\Sigma}_{Q,d})^{(\mathbf{G}^{\Sigma}_{Q,d},\chi_\theta)-s}. \tag{4.4}$$

However, the converse inclusion is not true in general (as Lemma 4.7 does not hold for stable points). Due to (4.4), we have a geometric quotient

$$\mathcal{M}^{(\Sigma,\theta)-rs}_{Q,d} := (\mathrm{Rep}^{\Sigma}_{Q,d})^{\chi_\theta-rs}/\mathbf{G}^{\Sigma}_{Q,d} = f^{-1}_\Sigma(\mathcal{M}^{\theta-gs}_{Q,d}),$$

which is open in $(\mathrm{Rep}^{\Sigma}_{Q,d})^{\chi_\theta-s}/\mathbf{G}^{\Sigma}_{Q,d} \simeq \mathcal{M}^{\tilde{\theta}-gs}_{Q/\Sigma,\tilde{d}}$ (using Corollary 4.6) and that we call the moduli space of (Σ,θ)-regularly stable representations of Q. Let

$$f^{rs}_\Sigma := f_\Sigma|_{\mathcal{M}^{(\Sigma,\theta)-rs}_{Q,d}} : \mathcal{M}^{(\Sigma,\theta)-rs}_{Q,d} \longrightarrow (\mathcal{M}^{\theta-gs}_{Q,d})^\Sigma$$

denote the restriction of f_Σ to the open subscheme $\mathcal{M}^{(\Sigma,\theta)-rs}_{Q,d} \subset \mathcal{M}^{\tilde{\theta}-gs}_{Q/\Sigma,\tilde{d}}$.

Proposition 4.10. *Let* $m = \mathbf{G}_{Q,d}(\Omega)\cdot M \in \mathcal{M}^{\theta-gs}_{Q,d}(\Omega)$ *for an algebraically closed field* Ω/k*. If non-empty, the geometric fibre* $(f^{rs}_\Sigma)^{-1}(m)$ *is in bijection with the pointed set* $\ker\big(H^1(\Sigma;\Delta(\Omega)) \longrightarrow H^1(\Sigma,\mathbf{G}_{Q,d}(\Omega))\big)$.

Proof. We omit the proof, as it is very similar to Proposition 3.3. □

Since Σ acts trivially on $\Delta \subset \mathbf{G}_{Q,d}$, we have $H^1(\Sigma;\Delta(\Omega)) \simeq H^1(\Sigma;\Omega^\times)$, where the latter is computed with respect to the trivial action of Σ on $\Omega^\times$. The next result gives a sufficient geometric condition for f^{rs}_Σ to be injective.

Corollary 4.11. *Suppose that* k *is algebraically closed and, for each* $\sigma \in \Sigma$*, there is a vertex* $v_\sigma \in V$ *such that* $\sigma(v_\sigma) = v_\sigma$*, then the morphism* f^{rs}_Σ *is injective.*

Proof. As k is algebraically closed by assumption, it suffices to check that f_Σ^{rs} is injective on closed points and for this we can use the description of the fibres given by Proposition 4.10. Thus, it suffices to show that $H^1(\Sigma;\Delta(k)) \longrightarrow H^1(\Sigma, \mathbf{G}_{Q,d}(k))$ is injective. By definition, an element in the kernel of that map is the cohomology class of a 1-cocycle $\alpha : \Sigma \longrightarrow \Delta$ of the form $\alpha(\sigma) = g\sigma(g^{-1})$ for some $g = (g_v)_{v\in V} \in \mathbf{G}_{Q,d}$. As α is Δ-valued, for each $\sigma \in \Sigma$, there exists $t_\sigma \in \mathbb{G}_m$ such that $(\alpha(\sigma)_v)_{v\in V} := (g_v g_{\sigma(v)}^{-1})_{v\in V} = (t_\sigma I_{d_v})_{v\in V}$. In particular, for $v = v_\sigma$, we have $t_\sigma I_{d_v} = g_v g_{\sigma(v)}^{-1} = g_v g_v^{-1} = I_{d_v}$; that is, $t_\sigma = 1$. Therefore, every such element α is trivial by our assumptions on the Σ-action on V. □

In this situation, we can also use a type map (as in Proposition 3.6) to determine whether f_Σ^{rs} is surjective.

Proposition 4.12. *Let Ω be an algebraically closed field containing k; then there is a map*

$$\mathcal{T}_\Omega : (\mathcal{M}_{Q,d}^{\theta-gs})^\Sigma(\Omega) \longrightarrow H^2(\Sigma, \Delta(\Omega)),$$

which we call the type map, such that $(\operatorname{Im} f_\Sigma^{rs})(\Omega) \subset \mathcal{T}_\Omega^{-1}([1])$.

Proof. The existence of $\mathcal{T}_\Omega$ is proved as in Proposition 3.6. To verify the final claim, we proceed as in the proof of Theorem 3.8. □

Example 4.13. If Σ is a cyclic group of order n and k is algebraically closed, then $H^2(\Sigma; \Delta(k)) \simeq H^2(\mathbb{Z}/n\mathbb{Z}; k^*) = k^*/(k^*)^{(n)} = \{1\}$.

In Example 4.23, we will see that one can have $(\operatorname{Im} f_\Sigma^{rs})(\Omega) \neq \mathcal{T}_\Omega^{-1}([1])$. The map f_Σ^{rs} is not surjective in general and, to account for that failure, we will alter the algebraic Σ-action on $\operatorname{Rep}_{Q,d}$ and $\mathbf{G}_{Q,d}$ by using a modifying family of elements in $\mathbf{G}_{Q,d}(k)$ in the following sense. We recall that Φ and Ψ denote the original Σ-actions on $\operatorname{Rep}_{Q,d}$ and $\mathbf{G}_{Q,d}$ respectively.

Definition 4.14 (Modifying family)**.** A modifying family of elements in $\mathbf{G}_{Q,d}(k)$ indexed by Σ is a tuple $u := (u_\sigma)_{\sigma\in\Sigma}$ of elements $u_\sigma \in \mathbf{G}_{Q,d}(k)$ indexed by $\sigma \in \Sigma$ such that

(1) $u_{1_\Sigma} = 1_{\mathbf{G}_{Q,d}(k)}$,
(2) For $\sigma_1, \sigma_2 \in \Sigma$, the elements $c_u(\sigma_1, \sigma_2) := u_{\sigma_1}\Psi_{\sigma_1}(u_{\sigma_2})u_{\sigma_1\sigma_2}^{-1} \in \mathbf{G}_{Q,d}(k)$ define a $\Delta(k)$-valued 2-cocycle c_u of Σ.

Lemma 4.15. *Let $u := (u_\sigma)_{\sigma\in\Sigma}$ be a modifying family of elements in $\mathbf{G}_{Q,d}(k)$ indexed by Σ. Then we can define modified Σ-actions*

$$\Phi^u : \Sigma \times \operatorname{Rep}_{Q,d} \longrightarrow \operatorname{Rep}_{Q,d}; \quad (\sigma, \phi) \longmapsto \Phi_\sigma^u(\phi) := u_\sigma \cdot \Phi_\sigma(\phi)$$

and

$$\Psi^u : \Sigma \times \mathbf{G}_{Q,d} \longrightarrow \mathbf{G}_{Q,d}; \quad (\sigma, g) \longmapsto \Psi_\sigma^u(g) := \operatorname{Ad}_{u_\sigma} \Psi_\sigma(g)$$

which are compatible in the sense of (4.2) *and such that the induced Σ-action on $\mathcal{M}_{Q,d}^{\theta-ss}$ coincides with that of Proposition 4.3.*

Proof. It is straightforward to check that Φ^u and Ψ^u are compatible Σ-actions. By using the universal property of the categorical quotient, as in Proposition 4.3, one proves there is an induced Σ-action on the quotient $\mathcal{M}_{Q,d}^{\theta-ss} = \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}$; this coincides with the Σ-action defined in loc. cit. as the following diagram commutes

$$\begin{array}{ccccc} \mathrm{Rep}_{Q,d}^{\chi_\theta-ss} & \xrightarrow{\Phi_\sigma} & \mathrm{Rep}_{Q,d}^{\chi_\theta-ss} & \xrightarrow{u_\sigma \cdot} & \mathrm{Rep}_{Q,d}^{\chi_\theta-ss} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi} & \\ \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d} & \xrightarrow{\Phi'_\sigma} & \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d} & & \end{array}$$

following the commutativity of Diagram (4.3). □

For a modifying family u, let $\mathrm{Rep}_{Q,d}^{\Sigma,u}$ and $\mathbf{G}_{Q,d}^{\Sigma,u}$ denote the fixed loci for the Σ-actions Φ^u and Ψ^u. Then $\mathrm{Rep}_{Q,d}^{\Sigma,u}$ is a closed subscheme of $\mathrm{Rep}_{Q,d}$ and $\mathbf{G}_{Q,d}^{\Sigma,u}$ is a closed subgroup of $\mathbf{G}_{Q,d}$, and moreover, the $\mathbf{G}_{Q,d}^{\Sigma,u}$-action preserves $\mathrm{Rep}_{Q,d}^{\Sigma,u}$.

Lemma 4.16. *For a modifying family $u = (u_\sigma)_{\sigma\in\Sigma}$ of elements in $\mathbf{G}_{Q,d}(k)$, the fixed locus for the u-modified Σ-action $\mathbf{G}_{Q,d}^{\Sigma,u}$ is a smooth connected reductive group.*

Proof. The fixed locus $\mathbf{G}_{Q,d}^{\Sigma,u}$ is the subgroup of elements $g = (g_v)_{v\in V}$ such that $g_v = u_{\sigma,v} g_{\sigma(v)} u_{\sigma,v}^{-1}$ for all $\sigma \in \Sigma$. If we pick representatives $v_1, \dots, v_r$ of the Σ-orbits in V, then

$$\mathbf{G}_{Q,d}^{\Sigma,u} \cong \prod_{i=1}^{r} C_{\mathbf{GL}_{d_{v_i}}}(\{u_{\sigma,v_i} : \sigma \in \Sigma, \sigma(v_i) = v_i\})$$

is isomorphic to a product of centraliser subgroups in general linear groups.

For any subset $S \subset \mathrm{Mat}_{n\times n}$, the centraliser $C_{\mathrm{Mat}_{n\times n}}(S)$ is a vector subspace of $\mathrm{Mat}_{n\times n}$, and thus is connected. For $S \subset \mathbf{GL}_n$, it follows that $C_{\mathbf{GL}_n}(S)$ is also connected, as this is the non-vanishing locus of a single polynomial (the determinant) in $C_{\mathrm{Mat}_{n\times n}}(S)$. Therefore, $\mathbf{G}_{Q,d}^{\Sigma,u}$ is connected. By [4, Proposition A.8.11], the fixed locus of any finite group scheme over k acting on any smooth k-scheme is smooth; hence, $\mathbf{G}_{Q,d}^{\Sigma,u}$ is smooth. Furthermore, by [4, Proposition A.8.12], for any linearly reductive group scheme H acting on a reductive group scheme G over k, the connected component of the identity of the fixed locus G^H with its reduced scheme structure is reductive; hence, it follows that $\mathbf{G}_{Q,d}^{\Sigma,u}$ is reductive. □

In fact, the above argument shows that for any finite subgroup Σ acting on a product of general linear groups G over k, the fixed locus is smooth connected and reductive. However, this statement is not true for an arbitrary reductive group scheme over k in positive characteristic, as it is not necessarily true that the fixed locus is connected.

We can now give an analogue of Propositions 4.8, 4.10 and 4.12 for the Σ-action given by a modifying family u.

Theorem 4.17. *Let $u := (u_\sigma)_{\sigma\in\Sigma}$ be a modifying family of elements in $\mathbf{G}_{Q,d}(k)$ indexed by Σ. Then*

$$(\mathrm{Rep}_{Q,d}^{\Sigma,u})^{(\mathbf{G}_{Q,d}^{\Sigma,u},\chi_\theta)-ss} = \mathrm{Rep}_{Q,d}^{\Sigma,u} \times_{\mathrm{Rep}_{Q,d}} \mathrm{Rep}_{Q,d}^{(\mathbf{G}_{Q,d},\chi_\theta)-ss} \tag{4.5}$$

and there is a morphism

$$f_{\Sigma,u} : \mathrm{Rep}_{Q,d}^{\Sigma,u} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma,u} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^{\Sigma}.$$

Furthermore, there is a moduli space of (Σ, u, θ)*-regularly stable representations of* Q *given by* $(\mathrm{Rep}_{Q,d}^{\Sigma,u})^{\chi_\theta - rs} := \mathrm{Rep}_{Q,d}^{\Sigma,u} \times_{\mathrm{Rep}_{Q,d}} \mathrm{Rep}_{Q,d}^{(\mathbf{G}_{Q,d},\chi_\theta)-s}$ *and*

$$(\mathrm{Rep}_{Q,d}^{\Sigma,u})^{\chi_\theta - rs}/\mathbf{G}_{Q,d}^{\Sigma,u} = f_{\Sigma,u}^{-1}(\mathcal{M}_{Q,d}^{\theta-gs})$$

and if we denote the restriction of $f_{\Sigma,u}$ *to this open subscheme by*

$$f_{\Sigma,u}^{rs} : (\mathrm{Rep}_{Q,d}^{\Sigma,u})^{\chi_\theta - rs}/\mathbf{G}_{Q,d}^{\Sigma,u} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-gs})^{\Sigma},$$

then the image of $f_{\Sigma,u}^{rs}$ *is a closed subscheme of* $(\mathcal{M}_{Q,d}^{\theta-gs})^{\Sigma}$*. Non-empty fibres of* $f_{\Sigma,u}^{rs}$ *are in bijection with the pointed set*

$$\ker\left(H^1(\Sigma;\Delta(\Omega)) \longrightarrow H^1_u(\Sigma,\mathbf{G}_{Q,d}(\Omega))\right)$$

where Σ *acts on* $\mathbf{G}_{Q,d}$ *via the action* Ψ^u *defined in Lemma 4.15. Moreover, for all algebraically closed fields* Ω *containing* k*, we have* $\mathrm{Im}\, f_{\Sigma,u}^{rs}(\Omega) \subset \mathcal{T}_\Omega^{-1}([c_u])$*, where* $[c_u]$ *is the cohomology class of by the* $\Delta(\Omega)$*-valued 2-cocycle* $c_u(\sigma_1,\sigma_2) := u_{\sigma_1}\Psi_{\sigma_1}(u_{\sigma_2})u_{\sigma_1\sigma_2}^{-1}$ *on* Σ *defined by* u*.*

Proof. The result follows from simple modifications of the arguments given in the proofs of Propositions 4.8, 4.10 and 4.12. □

4.3. **A decomposition of the fixed locus.** In this section, we let k be an algebraically closed field (see Remark 4.22 for results over an arbitrary field k). Our goal is to give a description of the fixed locus $(\mathcal{M}_{Q,d}^{\theta-gs})^{\Sigma}$ for the action of $\Sigma \subset \mathrm{Aut}^+(Q)$ in terms of the images of morphisms $f_{\Sigma,u}^{rs}$ given by modifying families u (*cf.* Theorem 4.17). We will do this in two stages: first by describing a given fibre of the type map $\mathcal{T}$ as a disjoint union of images of such morphisms, and then by taking the union over all fibres of $\mathcal{T}$ in order to produce a decomposition of the fixed locus $(\mathcal{M}_{Q,d}^{\theta-s})^{\Sigma}$. Finally, we will illustrate this decomposition with some simple examples. Before we can give the decomposition result for a fibre of the type map, we need a few lemmas.

Lemma 4.18. *Let* u' *be a modifying family of elements of* $\mathbf{G}_{Q,d}(k)$ *such that* $c_{u'}$ *is cohomologous to a* $\Delta(k)$*-valued 2-cocycle* c*, then there is a family* $(a'_\sigma)_{\sigma\in\Sigma}$ *of elements of* Δ *such that* $u''_\sigma := a'_\sigma u'_\sigma$ *is a modifying family of elements in* $\mathbf{G}_{Q,d}(k)$ *with*

1. $\Phi_\sigma^{u''} = \Phi_\sigma^{u'}$ *and* $\Psi_\sigma^{u''} = \Psi_\sigma^{u'}$ *for all* $\sigma \in \Sigma$*;*
2. $c_{u''} = c$ *as* Δ*-valued 2-cocycles.*

Proof. Since $c_{u'}$ and c are cohomologous, there exists a family $(a'_\sigma)_{\sigma\in\Sigma}$ of elements of Δ such that

$$c(\sigma_1,\sigma_2)c_{u'}(\sigma_1,\sigma_2)^{-1} = a'_{\sigma_1}\sigma_1(a'_{\sigma_2})(a'_{\sigma_1\sigma_2})^{-1} \quad \text{for } \sigma\in\Sigma, i=1,2. \tag{4.6}$$

Let $u''_\sigma := a'_\sigma u'_\sigma$, then $\Phi_\sigma^{u''} = \Phi_\sigma^{u'}$ and $\Psi_\sigma^{u''} = \Psi_\sigma^{u'}$, because $a'_\sigma \in \Delta$. Since Δ is central and fixed by the Σ-action, it follows from (4.6) that $c_{u''} = c$. □

For a modifying family u, let $Z_u^1(\Sigma,\mathbf{G}_{Q,d}(k))$ denote the set of $\mathbf{G}_{Q,d}(k)$-valued normalised 1-cocycles on Σ, calculated with respect to the Σ-action given by Ψ^u. The following lemma describes which modifying families give the same $\Delta(k)$-valued 2-cocycle.

Lemma 4.19. *Let $u = (u_\sigma)_{\sigma\in\Sigma}$ be a modifying family of elements in $\mathbf{G}_{Q,d}(k)$ indexed by Σ. Then there is a bijection*

$$\begin{array}{ccc} \{\text{modifying families } u' \mid c_{u'} = c_u\} & \longleftrightarrow & Z^1_u(\Sigma, \mathbf{G}_{Q,d}(k)) \\ u' = (u'_\sigma)_{\sigma\in\Sigma} & \longmapsto & (b_{u'} : \Sigma \longrightarrow \mathbf{G}_{Q,d}(k),\ \sigma \longmapsto u'_\sigma u_\sigma^{-1}) \\ u^b = (u^b_\sigma := b(\sigma)u_\sigma)_{\sigma\in\Sigma} & \longleftarrow & (b : \Sigma \longrightarrow \mathbf{G}_{Q,d}(k),\ \sigma \longmapsto b(\sigma)). \end{array}$$

Proof. For a modifying family u' with $c_u = c_{u'}$, we check that $b_{u'}$ is a 1-cocycle:

$$\begin{aligned} b_{u'}(\sigma_1\sigma_2) = u'_{\sigma_1\sigma_2}u^{-1}_{\sigma_1\sigma_2} &= \big(c_{u'}(\sigma_1,\sigma_2)^{-1}u'_{\sigma_1}\sigma_1(u'_{\sigma_2})\big)\big(\sigma_1(u^{-1}_{\sigma_2})u^{-1}_{\sigma_1}c_u(\sigma_1,\sigma_2)\big) \\ &= (u'_{\sigma_1}u^{-1}_{\sigma_1})(u_{\sigma_1}\sigma_1(u'_{\sigma_2}u^{-1}_{\sigma_2})u^{-1}_{\sigma_1}) = b_{u'}(\sigma_1)\Psi^u_{\sigma_1}(b_{u'}(\sigma_2)), \end{aligned}$$

as $c_u = c_{u'}$ is valued in the central subgroup $\Delta(k) \subset \mathbf{G}_{Q,d}(k)$.

For $b \in Z^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$, we check that u^b is a modifying family with $c_{u^b} = c_u$:

$$\begin{aligned} c_{u^b}(\sigma_1,\sigma_2) &:= u^b_{\sigma_1}\sigma_1(u^b_{\sigma_2})(u^b_{\sigma_1\sigma_2})^{-1} = b(\sigma_1)u_{\sigma_1}\sigma_1(b(\sigma_2)u_{\sigma_2})(b(\sigma_1\sigma_2)u_{\sigma_1\sigma_2})^{-1} \\ &= b(\sigma_1)\underbrace{(u_{\sigma_1}\sigma_1(b(\sigma_2))u^{-1}_{\sigma_1})}_{=\Psi^u_{\sigma_1}(b(\sigma_2))}\underbrace{u_{\sigma_1}\sigma_1(u_{\sigma_2})u^{-1}_{\sigma_1\sigma_2}}_{=c_u(\sigma_1,\sigma_2)\in\Delta}b(\sigma_1\sigma_2)^{-1} \\ &= c_u(\sigma_1,\sigma_2). \end{aligned}$$

This completes the proof, as it is clear that these two maps are inverse to each other. □

Note that two 1-cocycles $b_{u'}$ and $b_{u''}$ are cohomologous (that is, there exists $g \in \mathbf{G}_{Q,d}(k)$ such that, for all $\sigma \in \Sigma$, $b_{u''}(\sigma) = gb_{u'}(\sigma)\Psi^u_\sigma(g^{-1})$) if and only if there exists $g \in \mathbf{G}_{Q,d}(k)$ such that, for all $\sigma \in \Sigma$, $u''_\sigma = gu'_\sigma\sigma(g^{-1})$.

Remark 4.20. Let u be a modifying family. Suppose that $b_1, b_2 \in Z^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$ are cohomologous; then we have morphisms $f^{rs}_{\Sigma,u^{b_i}}$ (*cf.* Theorem 4.17) and the images of these morphisms coincide: $\operatorname{Im} f^{rs}_{\Sigma,u^{b_1}} = \operatorname{Im} f^{rs}_{\Sigma,u^{b_2}}$, as the action of g gives an isomorphism $\mathrm{Rep}^{\Sigma,u^{b_1}}_{Q,d} \xrightarrow{\simeq} \mathrm{Rep}^{\Sigma,u^{b_2}}_{Q,d}$.

For a modifying family u, we can now give a decomposition of the fibre of the type map $\mathcal{T} : (\mathcal{M}^{\theta-s}_{Q,d})^\Sigma(k) \longrightarrow H^2(\Sigma, \Delta(k))$ over $[c_u]$, where the indexing set is the orbit space $H^1_u(\Sigma, \mathbf{G}_{Q,d}(k))/H^1(\Sigma, \Delta(k))$: as $\Delta(k)$ is central and Σ-invariant in $\mathbf{G}_{Q,d}(k)$, the map $((\delta, u) \longmapsto (\delta u)_\sigma := (\delta_\sigma u_\sigma)_{\sigma\in\Sigma}$ indeed induces an action of the group $H^1_u(\Sigma, \Delta(k))$ on the set $H^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$; moreover, $H^1_u(\Sigma, \Delta(k))$ is actually independent of the modifying family u. For $[b] \in H^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$, we shall denote its $H^1(\Sigma, \Delta(k))$-orbit by $\overline{[b]}$.

Theorem 4.21. *Let u be a modifying family of elements in $\mathbf{G}_{Q,d}(k)$ indexed by Σ; then there is a decomposition*

$$\mathcal{T}^{-1}([c_u]) = \bigsqcup_{\overline{[b]}\in H^1_u(\Sigma,\mathbf{G}_{Q,d}(k))/H^1(\Sigma,\Delta(k))} (\operatorname{Im} f^{rs}_{\Sigma,u^b})(k),$$

where u^b is the modifying family determined by u and a choice of 1-cocycle $b \in [b]$. Furthermore, the non-empty fibres of f^{rs}_{Σ,u^b} are described by Theorem 4.17.

Proof. For each $[b] \in H^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$, we take a representative b of $[b]$ and consider the morphism f^{rs}_{Σ,u^b} as in Theorem 4.17; the image of this morphism does not depend on our choice of representative by Remark 4.20.

To show that these images cover $\mathcal{T}^{-1}([c_u])$, take $\mathbf{G}_{Q,d} \cdot M \in \mathcal{T}^{-1}([c_u])$, so by definition of the type map, there exists a modifying family $u' = (u'_\sigma)_{\sigma\in\Sigma}$ of elements in $\mathbf{G}_{Q,d}(k)$ such that $M \in \mathrm{Rep}_{Q,d}^{\Sigma,u'}$ and $[c_{u'}] = [c_u]$. By Lemma 4.18, we can assume that $c_{u'} = c_u$ and so, by Lemma 4.19, there exists $b \in Z^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$ such that $u' = u^b$. Hence, $\mathbf{G}_{Q,d} \cdot M \in \operatorname{Im} f^{rs}_{\Sigma,u^b}$.

To prove that this union is disjoint, suppose that $\mathbf{G}_{Q,d}\cdot M \in \operatorname{Im} f^{rs}_{\Sigma,u^{b_i}}$ for $i = 1, 2$. Then there exist modifying families u_i, for $i = 1, 2$, such that

(1) $[b_{u_i}] = [b_i] \in H^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$,
(2) $M \in \mathrm{Rep}_{Q,d}^{\Sigma,u_i}$ for $i = 1, 2$,
(3) $c_{u_1} = c_{u_2} = c_u$.

The only one of these assertions which is not clear is the final one, which follows from the fact that $c_{u^{b_i}} = c_u$, for $i = 1, 2$, and the observation that if $[b] = [b']$, then $c_{u^b} = c_{u^{b'}}$. From (2), we deduce that $a_\sigma := u_{2,\sigma}u_{1,\sigma}^{-1} \in \mathrm{Stab}_{\mathbf{G}_{Q,d}}(M) = \Delta(k)$, from which we conclude that $b_{u_2,\sigma} = a_\sigma b_{u_1,\sigma}$ for all σ, therefore that $[b_{u_1}]$ and $[b_{u_2}]$ lie in the same $H^1(\Sigma, \Delta(k))$-orbit in $H^1_u(\Sigma, \mathbf{G}_{Q,d}(k))$. This completes the proof. □

By definition of the type map $\mathcal{T} : (\mathcal{M}^{\theta-s}_{Q,d})^\Sigma(k) \longrightarrow H^2(\Sigma, \Delta(k))$, if $[c] \in \operatorname{Im}\mathcal{T}$, there exists a modifying family u with $[c] = [c_u]$. We can now prove Theorem 1.2.

Proof of Theorem 1.2. We have a decomposition

$$(\mathcal{M}^{\theta-s}_{Q,d})^\Sigma(k) = \bigsqcup_{[c]\in\operatorname{Im}\mathcal{T}} \mathcal{T}^{-1}([c])$$

and, for each $[c] \in \operatorname{Im}\mathcal{T}$, there exists a modifying family u such that $[c] = [c_u]$. Hence, by Theorem 4.21 we obtain the above decomposition on the level of k-points. Since $\operatorname{Im} f^{rs}_{\Sigma,u^b}$ and $(\mathcal{M}^{\theta-s}_{Q,d})^\Sigma$ are varieties over the algebraically closed field k, this set-theoretic decomposition on the level of closed points gives a decomposition as varieties. The fibres of the morphisms f^{rs}_{Σ,u^b} are described as in Theorem 4.21. □

Remark 4.22. We make the following observations on Theorem 1.2.

(1) We note that, for k algebraically closed, $\mathcal{M}^{\theta-gs}_{Q,d}$ is smooth, as it can be considered as a geometric quotient of the smooth k-variety $\mathrm{Rep}^{\chi_\theta-s}_{Q,d}$ by the free action of the reductive group $\overline{\mathbf{G}}_{Q,d} := \mathbf{G}_{Q,d}/\Delta$ and so this quotient is smooth by Luna's étale slice Theorem. Consequently, $(\mathcal{M}^{\theta-gs}_{Q,d})^\Sigma$ is smooth by [4, Proposition A.8.11].
(2) If k is not algebraically closed, then we obtain a set-theoretic decomposition of $(\mathcal{M}^{\theta-gs}_{Q,d})^\Sigma(\Omega)$ for any algebraically closed field Ω with $k \subset \Omega$ via the same techniques.
(3) Over the complex numbers $\mathcal{M}^{\theta-s}_{Q,d}$ is a smooth Kähler manifold, as it is homeomorphic, by the Kempf-Ness Theorem [18], to the Kähler reduction of the complex vector space $\mathrm{Rep}_{Q,d}$ (by the action of the maximal compact subgroup of $\mathbf{G}_{Q,d}$). The group Σ acts algebraically on $\mathcal{M}^{\theta-s}_{Q,d}$, therefore preserves the complex structure. In particular, $(\mathcal{M}^{\theta-s}_{Q,d})^\Sigma$ is a holomorphic, hence Kähler, submanifold of $\mathcal{M}^{\theta-s}_{Q,d}$.

We end this section with two examples that illustrate the decomposition theorem.

Example 4.23. Let Q be the quiver $1\bullet \underset{b}{\overset{a}{\rightrightarrows}} \bullet 2$ with covariant involution σ which fixes the vertex set $V = \{1, 2\}$ and sends arrow a to b. As all dimension vectors d and stability parameters θ are σ-compatible, we choose $d = (1, 1)$ and $\theta = (1, -1)$. Then $(s_1, s_2) \in \mathbf{G}_{Q,d} = \mathbb{G}_m^2$ acts on $M = (M_a, M_b) \in \mathrm{Rep}_{Q,d} = \mathbb{A}^2$ by

$$(s_1, s_2) \cdot (M_a, M_b) = (s_2 M_a s_1^{-1}, s_2 M_b s_1^{-1}).$$

Let us write $\mathbb{A}^2 = \operatorname{Spec} k[X_a, X_b]$; then both X_a and X_b are semi-invariant functions for the character χ_θ defined at (2.3). Hence, we have the GIT quotient

$$\pi : \mathrm{Rep}_{Q,d}^{\theta-ss} = \mathbb{A}^2 - \{0\} \longrightarrow \mathcal{M}_{Q,d}^{\theta-ss} := \mathrm{Rep}_{Q,d} /\!/_{\chi_\theta} \mathbf{G}_{Q,d} = \operatorname{Proj} k[X_a, X_b] = \mathbb{P}^1.$$

In fact, the θ-semistable locus and θ-stable locus agree, so that this is a geometric quotient, and, moreover, every GIT semistable orbit has stabiliser group $\Delta(k)$. The involution σ on Q induces an involution Φ_σ on $\mathrm{Rep}_{Q,d}$ given by

$$\Phi_\sigma((M_a, M_b)) = (M_b, M_a)$$

and induces a trivial action on $\mathbf{G}_{Q,d}$, as σ fixes the vertex set. The induced involution Φ'_σ on $\mathcal{M}_{Q,d}^{\theta-ss} = \mathbb{P}^1$ is given by $\Phi'_\sigma([M_a : M_b]) = [M_b : M_a]$ and the fixed locus is $(\mathbb{P}^1)^\sigma = \{[1 : 1], [1 : -1]\}$. The GIT quotient of $\mathbf{G}_{Q,d}^\sigma = \mathbf{G}_{Q,d}$ acting on $\mathrm{Rep}_{Q,d}^\sigma = \{(M_a, M_b) : M_a = M_b\} \cong \mathbb{A}^1$ with respect to χ_θ is

$$(\mathrm{Rep}_{Q,d}^\sigma)^{\theta-ss} \cong \mathbb{A}^1 - \{0\} \longrightarrow \mathrm{Rep}_{Q,d}^\sigma /\!/_{\chi_\theta} \mathbf{G}_{Q,d} = \operatorname{Proj} k[X_a] = \operatorname{Spec} k,$$

which we can interpret this quotient as a moduli space for θ-semistable representations of dimension $\tilde{d} = (1, 1)$ of the quotient quiver $Q/\sigma = \bullet \longrightarrow \bullet$ by Corollary 4.6. The morphism

$$f_\sigma : \mathrm{Rep}_{Q,d}^\sigma /\!/_{\chi_\theta} \mathbf{G}_{Q,d} \cong \operatorname{Spec} k \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^\sigma = \{[1 : 1], [1 : -1]\}$$

is injective but, when $\mathrm{char}(k) \neq 2$, it is not surjective, as its image equal to the point $[1 : 1]$. Let u be the modifying family given by $u_\sigma = (1, -1) \in \mathbf{G}_{Q,d}(k)$; then we can modify the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathrm{Rep}_{Q,d}$ by

$$\Phi_\sigma^u((M_a, M_b)) := u_\sigma \cdot \Phi_\sigma((M_a, M_b)) = (-M_b, -M_a)$$

and the modified action on $\mathbf{G}_{Q,d}$ remains trivial. Then the GIT quotient of $\mathbf{G}_{Q,d}^{\sigma,u} = \mathbf{G}_{Q,d}$ acting on $\mathrm{Rep}_{Q,d}^{\sigma,u} = \{(M_a, M_b) : M_a = -M_b\} \cong \mathbb{A}^1$ with respect to χ_θ is a point and the image of $f_{\sigma,u} : \mathrm{Rep}_{Q,d}^{\sigma,u} /\!/_{\chi_\theta} \mathbf{G}_{Q,d} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^\sigma$ is the point $[1 : -1] \in (\mathbb{P}^1)^\sigma$. Hence, the fixed locus decomposes into two pieces

$$(\mathcal{M}_{Q,d}^{\theta-ss})^\sigma = \{[1 : 1], [1 : -1]\} \cong \mathrm{Rep}_{Q,d}^\sigma /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^\sigma \bigsqcup \mathrm{Rep}_{Q,d}^{\sigma,u} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\sigma,u}. \tag{4.7}$$

Let us now explain this decomposition in terms of the group cohomology of Σ. First, the injectivity of f_σ follows from Proposition 4.10 and the fact that $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $\mathbf{G}_{Q,d}$, so that $H^1(\Sigma; \Delta(k)) \simeq \{a \in k \mid a^2 = 1_k\} = \{\pm 1_k\}$ and

$$H^1(\Sigma; \mathbf{G}_{Q,d}(k)) \simeq \{(s_1, s_2) \in \mathbb{G}_m(k) \times \mathbb{G}_m(k) \mid s_1^2 = s_2^2 = 1_k\} \simeq \{\pm 1_k\} \times \{\pm 1_k\}$$

and the map $H^1(\Sigma; \Delta(k)) \longrightarrow H^1(\Sigma; \mathbf{G}_{Q,d}(k))$ is conjugate to the group homomorphism $a \longmapsto (a, a)$, which is injective. There is only one fibre for the type map, because $H^2(\Sigma; \Delta(k)) = 1$ by Example 4.13. This fibre has two components, as

$$\begin{aligned} H^1(\Sigma; \mathbf{G}_{Q,d}(k))/H^1(\Sigma, \Delta(k)) &= \big(\{\pm 1_k\} \times \{\pm 1_k\}\big)/\{\pm 1\} \\ &= \{(1, 1), (1, -1)\}, \end{aligned}$$

which has two distinct elements if $\mathrm{char}(k) \neq 2$; this gives the decomposition (4.7).

Example 4.24. Let Q be the quiver $1\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet 2$ with covariant involution σ which sends vertex 1 to 2, and sends arrow a to b. Then a dimension vector $d = (d_1, d_2)$ is σ-compatible if and only if $d_1 = d_2$ and similarly for θ. For $\sum_i \theta_i d_i = 0$, we need $\theta = (0,0)$, if θ and d are both σ-compatible. Let $d = (1,1)$ and $\theta = (0,0)$; then $(s_1, s_2) \in \mathbf{G}_{Q,d} = \mathbb{G}_m^2$ acts on $(M_a, M_b) \in \mathrm{Rep}_{Q,d} = \mathbb{A}^2$ by

$$(s_1, s_2) \cdot (M_a, M_b) = (s_2 M_a s_1^{-1}, s_1 M_b s_2^{-1})$$

and the affine GIT quotient of this action is

$$\pi : \mathrm{Rep}_{Q,d} \longrightarrow \mathcal{M}_{Q,d}^{\theta-ss} := \mathrm{Rep}_{Q,d} /\!/ \mathbf{G}_{Q,d} = \mathrm{Spec}\, k[X_a X_b] \cong \mathbb{A}^1;$$

this restricts to a geometric quotient on the stable locus, which is the complement to the union of the coordinate axes. The action of σ on $\mathbf{G}_{Q,d}$ is given by $(s_1, s_2) \longmapsto (s_2, s_1)$ and so $\mathbf{G}_{Q,d}^{\sigma} = \Delta$, and the σ-action on $\mathrm{Rep}_{Q,d}$ is given by $(M_a, M_b) \longmapsto (M_b, M_a)$. Hence, the induced action of σ on $\mathcal{M}_{Q,d}^{\theta-ss} \cong \mathbb{A}^1$ is trivial.

The action of $\mathbf{G}_{Q,d}^{\sigma} = \Delta$ on $\mathrm{Rep}_{Q,d}^{\sigma} \cong \mathbb{A}^1$ is also trivial and so the affine GIT quotient of this action is the identity map on $\mathbb{A}^1$. We can interpret this as a moduli space for the quotient quiver Q/σ, which is the Jordan quiver with one vertex and a single loop.

Therefore, the morphism

$$f_\sigma : \mathrm{Rep}_{Q,d}^{\sigma} /\!/ \mathbf{G}_{Q,d}^{\sigma} \cong \mathbb{A}^1 /\!/ \mathbb{G}_m = \mathbb{A}^1 \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^{\sigma} = (\mathbb{A}^1)^{\sigma} = \mathbb{A}^1$$

is given by $z \longmapsto z^2$, which is not injective when $\mathrm{char}(k) \neq 2$. More concretely, we can see this on the level of representations. The two representations $M^{\pm} = (\pm 1, \pm 1) \in \mathrm{Rep}_{Q,d}^{\sigma}$ correspond to the same stable $\mathbf{G}_{Q,d}$-orbit (for example, if $g = (-1, 1) \in \mathbf{G}_{Q,d}$, then $g \cdot M^+ = M^-$). However, the $\mathbf{G}_{Q,d}^{\sigma}$-orbits of these points are distinct, as the $\mathbf{G}_{Q,d}^{\sigma}$-action is trivial.

To see the failure of injectivity on the level of group cohomology, we use Proposition 4.10 and the fact that $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbf{G}_{Q,d} = \mathbb{G}_m^2$ by swapping the two factors, so that $H^1(\Sigma; \Delta(k)) \simeq \{\pm 1_k\}$ and

$$H^1(\Sigma; \mathbf{G}_{Q,d}(k)) \simeq \frac{\{(s_1, s_2) \in \mathbb{G}_m(k) \times \mathbb{G}_m(k) \mid (s_1 s_2, s_2 s_1) = (1,1)\}}{\{(\frac{s_1}{s_2}, \frac{s_2}{s_1}) : (s_1, s_2) \in \mathbb{G}_m(k) \times \mathbb{G}_m(k)\}} \simeq \{(1,1)\},$$

which shows that f_{Σ}^{rs} is $2:1$ when $\mathrm{char}(k) \neq 2$.

Finally $H^2(\Sigma, \Delta(k)) = 1$ by Example 4.13, and $H^1(\Sigma; \mathbf{G}_{Q,d}(k)) = \{(1,1)\}$ so, by Theorem 4.21, one has $(\mathcal{M}_{Q,d}^{\theta-s})^{\sigma} = \mathcal{T}^{-1}([1]) = \mathrm{Im}\, f_{\Sigma}^{rs}$, as we saw above.

4.4. **Representation-theoretic interpretration.** Recall that k is a field and let $\mathcal{R}\mathrm{ep}_k(Q)$ denote the category of k-representations of Q.

Definition 4.25. A covariant (resp. contravariant) automorphism σ of Q determines a covariant (resp. contravariant) functor $\sigma : \mathcal{R}\mathrm{ep}_k(Q) \longrightarrow \mathcal{R}\mathrm{ep}_k(Q)$, which on a k-representation $W = ((W_v)_{v \in V}, (\varphi_a)_{a \in A})$ is given by

$$\sigma(W) = \begin{cases} ((W_{\sigma(v)})_{v \in V}, (\varphi_{\sigma(a)})_{a \in A}) & \text{if } \sigma \text{ is covariant,} \\ ((W_{\sigma(v)}^*)_{v \in V}, (\varphi_{\sigma(a)}^*)_{a \in A}) & \text{if } \sigma \text{ is contravariant.} \end{cases}$$

Note that if W has dimension $d = (d_v)_{v\in V}$ then $\sigma(W)$ has dimension $\sigma(d) = (d_{\sigma(v)})_{v\in V}$. In order for subrepresentations of $\sigma(W)$ to correspond canonically and bijectively to subrepresentations of W, we restrict ourselves to the covariant case $\Sigma < \mathrm{Aut}^+(Q)$ in this subsection.

Lemma 4.26. *Let $\theta = (\theta_v)_{v\in V}$ be a Σ-compatible stability parameter. Then, for all $\sigma \in \Sigma$, a k-representation W of Q is θ-(semi)stable if and only if $\sigma(W)$ is θ-(semi)stable.*

Proof. We note that for all k-representations W, we have, by Σ-compatibility of θ,

$$\mu_\theta(\sigma(W)) = \frac{\sum_{v\in V}\theta_v \dim W_{\sigma(v)}}{\sum_{v\in V}\dim W_{\sigma(v)}} = \frac{\sum_{v\in V}\theta_{\sigma^{-1}(v)}\dim W_v}{\sum_{v\in V}\dim W_v} = \mu_\theta(W).$$

As $W' \subset W$ is a subrepresentation if and only if $\sigma(W') \subset \sigma(W)$ is a subrepresentation, the statement is proved. ◻

Definition 4.27 (Equivariant representations)**.** For $\Sigma \subset \mathrm{Aut}^+(Q)$, let $\mathcal{R}\mathrm{ep}_k(Q,\Sigma)$ denote the category whose objects are pairs (W,γ) consisting of an object W of $\mathcal{R}\mathrm{ep}_k(Q)$ and a family of isomorphisms $(\gamma_\sigma : \sigma(W) \stackrel{\simeq}{\longrightarrow} W)_{\sigma\in\Sigma}$ such that, for all $(\sigma_1,\sigma_2) \in \Sigma\times\Sigma$, we have $\gamma_{\sigma_1\sigma_2} = \gamma_{\sigma_1}\sigma_1(\gamma_{\sigma_2})$, and whose morphisms are morphisms of representations that commute to the $(\gamma_\sigma)_{\sigma\in\Sigma}$.

There is a faithful forgetful functor $\mathcal{R}\mathrm{ep}_k(Q,\Sigma) \longrightarrow \mathcal{R}\mathrm{ep}_k(Q)$, and a representation of Q can only lie in the essential image of this functor, if its dimension vector is Σ-compatible. More generally, given a 2-cocycle $c \in Z^2(\Sigma; k^\times)$, we define (Σ, c)-equivariant representations of Q to be pairs (W,γ) as above, except that we now ask for $\gamma_{\sigma_1}\sigma_1(\gamma_{\sigma_2}) = c(\sigma_1,\sigma_2)\gamma_{\sigma_1\sigma_2}$. This defines a category $\mathcal{R}\mathrm{ep}_k(Q,\Sigma,c)$ analogously to the case of $c \equiv 1$. The following result is then easily checked.

Lemma 4.28. *If $u = (u_\sigma)_{\sigma\in\Sigma}$ is a modifying family in the sense of Definition 4.14 and c_u is the associated 2-cocycle, there is a bijection between isomorphism classes of d-dimensional objects of $\mathcal{R}\mathrm{ep}_k(Q,\Sigma,c_u)$ and the set $\mathrm{Rep}_{Q,d}^{\Sigma,u}(k)/\mathbf{G}_{Q,d}^{\Sigma,u}(k)$.*

The point of the above is that there is a natural notion of θ-semistability in $\mathcal{R}\mathrm{ep}_k(Q,\Sigma,c_u)$, that will eventually coincide with GIT semistability for the $\mathbf{G}_{Q,d}^{\Sigma,u}$-action on $\mathrm{Rep}_{Q,d}^{\Sigma,u}$ (with respect to the character $\chi_\theta|_{\mathbf{G}_{Q,d}^{\Sigma,u}}$) for Σ-compatible θ; see Theorem 4.33.

Definition 4.29 ((Σ,θ)-(semi)stability)**.** Let $c \in Z^2(\Sigma; k^\times)$. A (Σ, c)-equivariant representation (W,γ) of Q is called (Σ,θ)-(semi)stable if, for all non-zero proper subrepresentations (W',γ') in $\mathcal{R}\mathrm{ep}(Q,\Sigma,c)$, one has $\mu_\theta(W')(\leq)\mu_\theta(W)$.

We henceforth fix a Σ-compatible stability parameter θ.

Proposition 4.30. *Let (W,γ) be a (Σ,c)-equivariant representation of Q. Then the following are equivalent:*

1. *W is θ-semistable as a representation of Q.*
2. *(W,γ) is (Σ,θ)-semistable.*

Proof. Evidently, (1) implies (2). For the converse, we proceed by contrapositive using the uniqueness of the *scss* subrepresentation $U \subset W$ (Definition 2.3). So let us assume that W is not θ-semistable. Since subrepresentations of $\sigma(W)$ correspond bijectively to subrepresentations of W and θ is Σ-compatible, we have that $\sigma(U)$ is the *scss* of $\sigma(W)$ for each $\sigma \in \Sigma$. As $\gamma_\sigma : \sigma(W) \longrightarrow W$ is an isomorphism, $\gamma_\sigma(\sigma(U))$ is the *scss* of W, thus $\gamma_\sigma(\sigma(U)) = U$. In particular, $(U, \gamma|_U)$ is a (Σ, c)-equivariant subrepresentation of (W, γ), which can therefore not be (Σ, θ)-semistable. □

For $(W, \gamma) \in \mathcal{R}\mathrm{ep}_k(Q, \Sigma, c)$, one can construct a unique Harder–Narasimhan filtration of (W, γ) with respect to (Σ, θ)-semistability. Proposition 4.30 then has the following corollary.

Corollary 4.31. *Let (W, γ) be a (Σ, c)-equivariant representation of Q. Then the Harder–Narasimhan filtration of (W, γ) with respect to (Σ, θ)-semistability agrees with the Harder–Narasimhan filtration of W with respect to θ-semistability.*

Similarly to what we saw in Remark 2.5, the statement of Proposition 4.30 is no longer true if we replace semistability by stability.

Definition 4.32 ((Σ, θ)-regularly stable)**.** A (Σ, c)-equivariant k-representation (W, γ) is called (Σ, θ)-regularly stable if W is θ-geometrically stable as a k-representation (*cf.* Definition 2.6).

Theorem 4.33. *Let u be a modifying family and let c_u be the associated 2-cocycle. Let $M \in \mathrm{Rep}_{Q,d}^{\Sigma,u}$ and let (W, γ) be the corresponding (Σ, c_u)-equivariant representation of Q. Then:*

1. *M is $(\mathbf{G}_{Q,d}^{\Sigma,u}, \chi_\theta)$-semistable in the GIT sense if and only if (W, γ) is (Σ, θ)-semistable as a (Σ, c_u)-equivariant representation.*
2. *M is (Σ, u, χ_θ)-regularly stable in the sense of Definition 4.9 if and only if (W, γ) is (Σ, θ)-regularly stable in the sense of Definition 4.32.*

Proof. By Lemma 4.7, $(\mathbf{G}_{Q,d}^{\Sigma,u}, \chi_\theta)$-semistability of M is equivalent to $(\mathbf{G}_{Q,d}, \chi_\theta)$-semistability of that point. By Proposition 2.11, the latter is equivalent to θ-semistability of W. And, by Proposition 4.30, the latter is equivalent to (Σ, θ)-semistability of (W, γ) as (Σ, c_u)-equivariant representation. This proves (1). The proof of (2) follows along the same lines, in view of Defintions 4.9 and 4.32. □

We thus obtain a modular interpretation of the GIT quotients forming the domains of the morphisms $f_{\Sigma,u}$ and $f_{\Sigma,u}^{rs}$ introduced in Theorem 4.17. The proof is similar to Corollary 2.12.

Corollary 4.34. *The GIT quotient*

$$\mathcal{M}_{Q,d}^{(\Sigma,u,\theta)-ss} := \mathrm{Rep}_{Q,d}^{\Sigma,u} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma,u} \quad (\text{resp. } \mathcal{M}_{Q,d}^{(\Sigma,u,\theta)-rs} := (\mathrm{Rep}_{Q,d}^{\Sigma,u})^{\chi_\theta - rs} / \mathbf{G}_{Q,d}^{\Sigma,u})$$

is a coarse moduli space for (Σ, θ)-semistable (resp. (Σ, θ)-regularly stable) (Σ, c_u)-equivariant d-dimensional k-representations of Q.

The $\overline{k}$-points of $\mathcal{M}_{Q,d}^{(\Sigma,u,\theta)-rs}$ are isomorphism classes of (Σ, θ)-regularly stable (Σ, c_u)-equivariant d-dimensional $\overline{k}$-representations of Q and, as in Corollary 2.12, we can interpret the $\overline{k}$-points of $\mathcal{M}_{Q,d}^{(\Sigma,u,\theta)-ss}$ as S-equivalence classes of (Σ, θ)-semistable (Σ, c_u)-equivariant d-dimensional representations of Q by noting that every (Σ, θ)-semistable (Σ, c)-equivariant representation has a Jordan-Hölder filtration by (Σ, c)-equivariant subrepresentations whose successive quotients are (Σ, θ)-stable; this Jordan–Hölder filtration is not necessarily unique but the associated graded object is and we say that two (Σ, θ)-semistable (Σ, c_u)-equivariant representations are S-equivalent if the associated graded objects for their respective

Jordan–Hölder filtrations are isomorphic as (Σ, c_u)-equivariant representations; the desired modular interpretation of $\overline{k}$-points of the GIT quotient $\mathrm{Rep}_{Q,d}^{\sigma,u} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma,u}$ then follows by the same arguments as in Corollary 2.12, using again the results of [19] to relate S-equivalence in the representation-theoretic sense to equivalence of semistable orbits in the GIT setting.

As a consequence, when k is algebraically closed, we can revisit Theorem 1.2 as follows. For the Σ-action on $\mathcal{M}_{Q,d}^{\theta-s}$, there is a decomposition

$$(\mathcal{M}_{Q,d}^{\theta-s})^{\Sigma} = \bigsqcup_{\substack{[c_u]\in \mathrm{Im}\,\mathcal{T} \\ \overline{[b]}\in H^1_u(\Sigma,\mathbf{G}_{Q,d}(k))/H^1(\Sigma,\Delta(k))}} f_{\Sigma,u}^{rs}\left(\mathcal{M}_{Q,d}^{(\Sigma,u^b,\theta)-rs}\right)$$

where u^b is a modifying family determined by $[b]$ and u. If $H^1(\Sigma, \Delta(k)) = 1$, then

$$(\mathcal{M}_{Q,d}^{\theta-s})^{\Sigma} \cong \bigsqcup_{\substack{[c_u]\in \mathrm{Im}\,\mathcal{T} \\ \overline{[b]}\in H^1_u(\Sigma,\mathbf{G}_{Q,d}(k))/H^1(\Sigma,\Delta(k))}} \mathcal{M}_{Q,d}^{(\Sigma,u^b,\theta)-rs},$$

as one can deduce that the morphisms f_{Σ,u^b}^{rs} are all closed immersions, by using the fact that $H^1(\Sigma, \Delta(k)) = 1$ and Theorem 4.17.

4.5. **Actions by arbitrary groups of quiver automorphisms.** We now consider a subgroup $\Sigma \subset \mathrm{Aut}(Q)$ that contains at least one contravariant automorphism, as otherwise $\Sigma \subset \mathrm{Aut}^+(Q)$ and this is studied above. By restricting the homomorphism sign : $\mathrm{Aut}(Q) \longrightarrow \{\pm 1\}$ to Σ, we obtain a short exact sequence

$$1 \longrightarrow \Sigma^+ \longrightarrow \Sigma \longrightarrow \{\pm 1\} \longrightarrow 1,$$

where $\Sigma^+ \subset \mathrm{Aut}^+(Q)$.

Definition 4.35. For $\Sigma \subset \mathrm{Aut}(Q)$, we make the following definitions.

(1) A dimension vector $d = (d_v)_{v\in V}$ is Σ-compatible if $\sigma(d) = d$ for all $\sigma \in \Sigma$.
(2) A stability parameter $\theta = (\theta_v)_{v\in V}$ is Σ-compatible if $\sigma(\theta) = \mathrm{sign}(\sigma)\theta$ for all $\sigma \in \Sigma$.

For a Σ-compatible dimension vector d, we can construct induced actions of Σ on $\mathrm{Rep}_{Q,d}$ and $\mathbf{G}_{Q,d}$ as follows: for $\sigma \in \Sigma$, we define automorphisms

$$\Phi_\sigma : \mathrm{Rep}_{Q,d} \longrightarrow \mathrm{Rep}_{Q,d}, \quad (M_a)_{a\in A} \longmapsto \begin{cases} (M_{\sigma(a)})_{a\in A} & \text{if } \mathrm{sign}(\sigma) = 1, \\ ({}^tM_{\sigma(a)})_{a\in A} & \text{if } \mathrm{sign}(\sigma) = -1, \end{cases}$$

and

$$\Psi_\sigma : \mathbf{G}_{Q,d} \longrightarrow \mathbf{G}_{Q,d}, \quad (g_v)_{v\in V} \longmapsto \begin{cases} (g_{\sigma(v)})_{v\in V} & \text{if } \mathrm{sign}(\sigma) = 1, \\ ({}^tg_{\sigma(v)}^{-1})_{v\in V} & \text{if } \mathrm{sign}(\sigma) = -1. \end{cases}$$

These Σ-actions are compatible with the $\mathbf{G}_{Q,d}$-action on $\mathrm{Rep}_{Q,d}$ in the sense that

$$\Phi_\sigma(g \cdot M) = \Psi_\sigma(g) \cdot \Phi_\sigma(M). \tag{4.8}$$

Let $Q/\Sigma^+ = (V/\Sigma^+, A/\Sigma^+, \tilde{h}, \tilde{t})$ denote the quotient quiver for the action of Σ^+ (*cf.* Definition 4.4). By assumption, there is a contravariant automorphism $\sigma \in \Sigma$. It follows that this contravariant automorphism σ induces a contravariant involution $\tilde{\sigma}$ of Q/Σ^+ such that $\Sigma/\Sigma^+ \cong < \tilde{\sigma} >$, where

$$\tilde{\sigma}(\Sigma^+ \cdot v) := \Sigma^+ \cdot \sigma(v) \quad \text{and} \quad \tilde{\sigma}(\Sigma^+ \cdot a) := \Sigma^+ \cdot \sigma(a).$$

Moreover, the induced dimension vector $\tilde{d}$ on Q/Σ^+ is $\tilde{\sigma}$-compatible. Hence, we obtain the following description of the Σ-fixed loci:

$$\mathrm{Rep}_{Q,d}^{\Sigma} = (\mathrm{Rep}_{Q,d}^{\Sigma^+})^{\Sigma/\Sigma^+} = \mathrm{Rep}_{Q/\Sigma^+,\tilde{d}}^{\tilde{\sigma}}$$

and $\mathbf{G}_{Q,d}^{\Sigma} = (\mathbf{G}_{Q,d}^{\Sigma^+})^{\Sigma/\Sigma^+} = \mathbf{G}_{Q/\Sigma^+,\tilde{d}}^{\tilde{\sigma}}$.

Lemma 4.36. *Let d and θ be Σ-compatible. Then the following statements hold.*

(1) *The Σ-action on $\mathrm{Rep}_{Q,d}$ preserves the GIT (semi)stable sets $\mathrm{Rep}_{Q,d}^{\chi_\theta-(s)s}$.*
(2) *There is an induced algebraic Σ-action on the moduli spaces $\mathcal{M}_{Q,d}^{\theta-(gs)s}$.*

Proof. The proof of first statement follows from Proposition 4.3 for covariant automorphism groups and [35, Lemma 2.1]. The proof of the second statement then follows from the universal property of the GIT quotient. □

Henceforth, we assume that d and θ are both Σ-compatible, so there is an induced Σ-action on $\mathcal{M}_{Q,d}^{\theta-ss}$. If we restrict to the Σ^+-action, then there is a morphism

$$f_{\Sigma^+} : \mathrm{Rep}_{Q,d}^{\Sigma^+} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma^+} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^{\Sigma}$$

by Proposition 4.8, and the domain of this morphism is isomorphic to the moduli space $\mathcal{M}_{Q/\Sigma^+,\tilde{d}}^{\tilde{\theta}-ss}$ by Corollary 4.6. Moreover, there is an induced action of the contravariant involution $\tilde{\sigma}$ of Q/Σ^+ on the domain of f_{Σ^+}, as both $\tilde{d}$ and $\tilde{\theta}$ are $\tilde{\sigma}$-compatible. Hence, by Proposition 4.38 below, there is a morphism

$$f_{\tilde{\sigma}} : \mathrm{Rep}_{Q/\Sigma^+,\tilde{d}}^{\tilde{\sigma}} /\!/_{\chi_{\tilde{\theta}}} \mathbf{G}_{Q/\Sigma^+,\tilde{d}}^{\tilde{\sigma}} \longrightarrow (M_{Q/\Sigma^+,\tilde{d}}^{\tilde{\theta}-ss})^{\overline{\sigma}}$$

and so we can define $f_\Sigma := f_{\Sigma^+} \circ f_{\tilde{\sigma}}$ and obtain the following result.

Proposition 4.37. *For $\Sigma \subset \mathrm{Aut}(Q)$, there is a morphism*

$$f_\Sigma : \mathrm{Rep}_{Q,d}^{\Sigma} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\Sigma} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^{\Sigma}.$$

Finally, let us prove the following result for contravariant involutions.

Proposition 4.38. *Let σ be a contravariant involution of a quiver Q and suppose d and θ are σ-compatible. Then $\mathbf{G}_{Q,d}^{\sigma}$ is a reductive group and the following are equivalent for $M \in \mathrm{Rep}_{Q,d}^{\sigma}$.*

(1) *M is GIT semistable for the $\mathbf{G}_{Q,d}$-action on $\mathrm{Rep}_{Q,d}$ with respect to the character $\chi_\theta : \mathbf{G}_{Q,d} \longrightarrow \mathbb{G}_m$.*
(2) *M is GIT semistable for the $\mathbf{G}_{Q,d}^{\sigma}$-action on $\mathrm{Rep}_{Q,d}^{\sigma}$ with respect to the restricted character $\chi_\theta : \mathbf{G}_{Q,d}^{\sigma} \longrightarrow \mathbb{G}_m$.*

Furthermore, there is an induced morphism $f_\sigma : \mathrm{Rep}_{Q,d}^{\sigma} /\!/_{\chi_\theta} \mathbf{G}_{Q,d}^{\sigma} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-ss})^{\sigma}$.

Proof. For the statement that $\mathbf{G}_{Q,d}^{\sigma}$ is reductive, see [35, §2.2], where it is proved that $\mathbf{G}_{Q,d}^{\sigma}$ is isomorphic to a product of orthogonal and general linear groups. The proof of these equivalences follows by Lemma 4.7, and the construction of f_σ follows as in Proposition 4.8. □

Analogously to Proposition 4.12, one can also prove that for $\Sigma \subset \mathrm{Aut}(Q)$ and any algebraically closed field Ω/k, there is a type map

$$\mathcal{T}_\Omega : \mathcal{M}_{Q,d}^{\theta-gs}(\Omega)^{\Sigma} \longrightarrow H^2(\Sigma, \Delta(\Omega))$$

and define modifying families in an analogous way to Definition 4.14 in order to produce a decomposition of the Σ-fixed locus.

For a contravariant involution σ of Q, we can study the morphism f_σ, or more precisely, its restriction to the regularly stable locus $f_\sigma^{rs} : \mathcal{M}_{Q,d}^{(\sigma,\theta)-rs} \longrightarrow (\mathcal{M}_{Q,d}^{\theta-gs})^\sigma$, by using the group cohomology of $\mathbb{Z}/2\mathbb{Z} \cong <\sigma>$. One can prove a straightforward generalisation of Proposition 4.10 for the fibres of f_σ^{rs}; thus, if k is algebraically closed, then f_σ is injective, as $H^1(\mathbb{Z}/2\mathbb{Z}, k^\times) = 1$ (we recall that σ acts on $k^\times$ by inversion now). One can also define modifying families, which are now given by a single element $u_\sigma \in \mathbf{G}_{Q,d}(k)$ such that $u_\sigma \Psi_\sigma(u_\sigma) \in \Delta(k)$, and use these families to modify the actions Φ and Ψ, without changing the action on $\mathcal{M}_{Q,d}^{\theta-gs}$; as $\mathbb{Z}/2\mathbb{Z}$ acts on $k^\times$ by inversion, we have $H^2(\mathbb{Z}/2\mathbb{Z}, k^\times) = \{\pm 1\}$, so there are only two possible cohomology classes for the 2-cocycle associated to a modifying family. This appears in [35] in different language, coming from physics: the duality structures in *loc. cit.* correspond to our modifying families. One can also provide a decomposition of $(\mathcal{M}_{Q,d}^{\theta-s})^\sigma$ by varying these modifying families using the cohomology of $\mathbb{Z}/2\mathbb{Z}$. One can give a representation-theoretic description of this decomposition as in §4.4, by defining notions of (σ,θ)-(semi)stability for (σ, c_u)-equivariant representations. Young refers to such equivariant representations as self-dual representation (*cf.* [35, Theorem 2.7] for an analogue of Theorem 4.33 in the contravariant setting).

One can thus provide a decomposition of $(\mathcal{M}_{Q,d}^{\theta-gs})^\Sigma$ for an arbitrary subgroup $\Sigma \subset \mathrm{Aut}(Q)$ using the group cohomology of Σ; however, we do not go through the details, as it is analogous to the case where $\Sigma \subset \mathrm{Aut}^+(Q)$.

5. Branes

Starting from a quiver Q, moduli spaces of representations of the doubled quiver $\overline{Q}$ (satisfying some relations) have a natural algebraic symplectic structure and we show that automorphisms of $\overline{Q}$ provide natural examples of Lagrangian and symplectic subvarieties. Over the complex numbers, these moduli spaces are hyperkähler when they are smooth and we can describe the fixed locus in the language of branes [16] as follows.

Definition 5.1. A brane in a hyperkähler manifold $(M, g, I, J, K, \omega_I, \omega_J, \omega_K)$ is a submanifold which is either holomorphic or Lagrangian with respect to each of the three Kähler structures on M. A brane is called of type A (respectively B) with respect to a given Kähler structure if it is Lagrangian (respectively holomorphic) for this Kähler structure. The type of the brane is encoded in a triple $T_I T_J T_K$, where $T_I = A$ or B is the type for the Kähler structure (g, I) and so on.

As $K = IJ$, there are 4 possible types of branes: BBB, BAA, ABA and AAB. We will show that we can construct each type of brane as a fixed locus of an involution. The study of branes in Nakajima quiver varieties has already been initiated in [6], where the authors use involutions such as complex conjugation, multiplication by -1 and transposition, to construct different branes. In the present section, we construct new examples coming from automorphisms of the quiver.

5.1. **The algebraic case.**

We assume throughout that k is a field of characteristic different from 2.

Definition 5.2 (Doubled quiver)**.** For a quiver $Q = (V, A, h, t)$, the doubled quiver is $\overline{Q} = (V, \overline{A}, h, t)$ where $\overline{A} = A \cup A^*$ for $A^* := \{a^* : h(a) \longrightarrow t(a)\}_{a \in A}$.

The central motivation for considering the doubled quiver is that

$$\mathrm{Rep}_{\overline{Q},d} = \mathrm{Rep}_{Q,d} \times \mathrm{Rep}^*_{Q,d} \cong T^* \mathrm{Rep}_{Q,d}$$

is an algebraic symplectic variety, with the Liouville symplectic form ω. Explicitly, if $M = (M_a, M_{a^*})_{a\in A}$ and $N = (N_a, N_{a^*})_{a\in A}$ are points in $\mathrm{Rep}_{\overline{Q},d}$, then

$$\omega(M,N) = \sum_{a\in A} \mathrm{Tr}(M_a N_{a^*} - M_{a^*} N_a). \tag{5.1}$$

The action of $\mathbf{G}_{\overline{Q},d} = \mathbf{G}_{Q,d}$ on $\mathrm{Rep}_{\overline{Q},d}$ is symplectic and there is an algebraic moment map $\mu : \mathrm{Rep}_{\overline{Q},d} \longrightarrow \mathfrak{g}^*_{Q,d}$, where $\mathfrak{g}_{Q,d}$ is the Lie algebra of $\mathbf{G}_{Q,d}$; explicitly, for $M \in \mathrm{Rep}_{\overline{Q},d}$ and $B \in \mathfrak{g}_{Q,d}$ we have

$$\mu(M)\cdot B = \sum_{a\in A} \mathrm{Tr}(M_{a^*}(B^{\#}_M)_a) = \sum_{a\in A} \mathrm{Tr}(M_{a^*}(B_{h(a)}M_a - M_a B_{t(a)})) \tag{5.2}$$

where $B^{\#}_M = (B_{h(a)}M_a - M_a B_{t(a)})_{a\in A}$ is the infinitesimal action of B on $(M_a)_{a\in A}$. The moment map is a $\mathbf{G}_{Q,d}$-equivariant morphism that satisfies the infinitesimal lifting property $d_M\mu(\eta)\cdot B = \omega(B^{\#}_M, \eta)$. By using the standard non-degenerate quadratic form $(B,C) \longmapsto \mathrm{Tr}({}^tBC)$ on the Lie algebra of each general linear group, we can naturally identify $\mathfrak{g}_{Q,d} \cong \mathfrak{g}^*_{Q,d}$ and view the moment map as a morphism $\mu : \mathrm{Rep}_{\overline{Q},d} \longrightarrow \mathfrak{g}_{Q,d}$ given by $\mu(M) = \sum_{a\in A}[M_a, M_{a^*}]$.

Definition 5.3. Let χ be a character of $\mathbf{G}_{Q,d}$ and let $\eta \in \mathfrak{g}_{Q,d}$ be a coadjoint fixed point; then $\mathbf{G}_{Q,d}$ acts on $\mu^{-1}(\eta)$ by the equivariance of the moment map. The algebraic symplectic reduction at (χ,η) is the GIT quotient $\mu^{-1}(\eta)/\!/_\chi \mathbf{G}_{Q,d}$.

If the GIT semistable and stable locus on $\mu^{-1}(\eta)$ with respect to χ agree, then $\mu^{-1}(\eta)/\!/_\chi \mathbf{G}_{Q,d}$ is an algebraic symplectic orbifold, whose form is induced by the Liouville form on $T^*\mathrm{Rep}_d(Q)$; this is an algebraic version of the Marsden–Weinstein Theorem [22] (for example, see [10]). If, moreover, G acts freely on the χ-semistable locus, then the algebraic symplectic reduction is smooth. For $\chi = \chi_\theta$, we have a closed immersion

$$\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d} \hookrightarrow \mathcal{M}^{\theta-ss}_{\overline{Q},d}.$$

Moreover, for a tuple of complex numbers $(\eta_v)_{v\in V}$, which determines an adjoint fixed point $\eta = (\eta_v \mathrm{Id}_{d_v})_{v\in V} \in \mathfrak{g}$, we have that $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ is the moduli space of θ-semistable d-dimensional representations of $\overline{Q}$ satisfying the relations

$$\mathcal{R}_\eta = \Big\{ \sum_{a\,|\,t(a)=v} M_a M_{a^*} - \sum_{a\,|\,h(a)=v} M_{a^*}M_a = \eta_v I_{d_v} \quad \forall v\in V \Big\}.$$

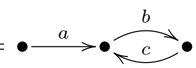
We now describe the fixed loci of quiver automorphism groups acting on this algebraic symplectic reduction in terms of its symplectic geometry.

Definition 5.4. We let $\mathrm{Aut}_*(\overline{Q})$ denote the subgroup of $\mathrm{Aut}(\overline{Q})$ consisting of automorphisms σ satisfying the conditions:

(1) For all $a \in A$, $\sigma(a^*) = \sigma(a)^*$.
(2) Either $\sigma(A) \subset A$ or $\sigma(A) \subset A^*$.

An automorphism $\sigma \in \mathrm{Aut}_*(\overline{Q})$ is said to be $\overline{Q}$-symplectic if $\sigma(A) \subset A$, and $\overline{Q}$-anti-symplectic if $\sigma(A) \subset A^*$.

Every $\sigma \in \mathrm{Aut}(Q)$ can be extended to a $\overline{Q}$-symplectic automorphism $\sigma \in \mathrm{Aut}_*(\overline{Q})$ by $\sigma(a^*) := \sigma(a)^*$. There is a canonical contravariant involution $\sigma \in \mathrm{Aut}_*(\overline{Q})$ which fixes all vertices and is given by $\sigma(a) := a^*$ on $a \in A$; it is $\overline{Q}$-anti-symplectic. Note that Condition (1) in Definition 5.4 does not imply Condition (2): consider for instance

$Q =$ 

and the contravariant involution $\sigma \in \mathrm{Aut}_*(\overline{Q})$ which fixes all vertices and sends $\sigma(a) = a^*$, $\sigma(b) = c$ and $\sigma(b^*) = c^*$.

There is a group morphism $s : \mathrm{Aut}_*(\overline{Q}) \longrightarrow \{\pm 1\}$ sending $\sigma \in \mathrm{Aut}_*(\overline{Q})$ to -1 if and only if σ is $\overline{Q}$-anti-symplectic.

Proposition 5.5. *Let $\sigma \in Aut_*(\overline{Q})$ and let d be a σ-compatible dimension vector. Then $\sigma^*\omega = s(\sigma)\,\omega$. Moreover, for $M \in \mathrm{Rep}_{\overline{Q},d}$ and $B \in \mathfrak{g}_{Q,d}$, we have $\mu(\sigma(M)) \cdot \sigma(B) = s(\sigma)(\mu(M) \cdot B)$.*

Proof. As d is σ-compatible, there is an induced action of σ on $\mathrm{Rep}_{\overline{Q},d}$. For $M \in \mathrm{Rep}_{\overline{Q},d}$ and let $Y, Z \in T_M \mathrm{Rep}_{\overline{Q},d} \cong \mathrm{Rep}_{\overline{Q},d}$, we have $d_M\sigma(Y) = (Y_{\sigma(a)})_{a \in \overline{A}}$ if σ is covariant, and $d_M\sigma(Y) = ({}^tY_{\sigma(a)})_{a\in\overline{A}}$ if σ is contravariant. As the calculations are similar in the covariant and contravariant case, we only give the details for a covariant automorphism σ, which is by assumption either $\overline{Q}$-symplectic or $\overline{Q}$-anti-symplectic. As $(\sigma^*\omega)_M(Y,Z) = \omega_{\sigma(M)}(d_M\sigma(Y), d_M\sigma(Z))$, we have

$$\begin{aligned}(\sigma^*\omega)_M(Y,Z) &= \sum_{a\in A} \mathrm{Tr}(Y_{\sigma(a)}Z_{\sigma(a^*)} - Y_{\sigma(a^*)}Z_{\sigma(a)}) \\ &= \sum_{\substack{a\in A\,|\\ b:=\sigma(a)\in A}} \mathrm{Tr}(Y_bZ_{b^*} - Y_{b^*}Z_b) - \sum_{\substack{a\in A\,|\\ c:=\sigma(a)^*\in A}} \mathrm{Tr}(Y_cZ_{c^*} - Y_{c^*}Z_c) \\ &= s(\sigma)\,\omega_M(Y,Z)\end{aligned}$$

where in the second equality we use the fact that $\sigma(a^*) = \sigma(a)^*$.

The derivative of the automorphism σ on $\mathbf{G}_{Q,d}$ induces an automorphism

$$\sigma : \mathfrak{g}_{Q,d} \longrightarrow \mathfrak{g}_{Q,d}, \quad (B_v)_{v\in V} \longmapsto \begin{cases} (B_{\sigma(v)})_{v\in V} & \text{if } \sigma \text{ is covariant,} \\ (-{}^tB_{\sigma(v)})_{v\in V} & \text{if } \sigma \text{ is contravariant.}\end{cases}$$

Thus, for covariant σ and for $M \in \mathrm{Rep}_{\overline{Q},d}$ and $B \in \mathfrak{g}_{Q,d}$, we have

$$\begin{aligned}\mu(\sigma(M))\cdot\sigma(B) &= \sum_{a\in A}\mathrm{Tr}(M_{\sigma(a)}M_{\sigma(a^*)}B_{\sigma(h(a))} - B_{\sigma(t(a))}M_{\sigma(a^*)}M_{\sigma(a)}) \\ &= \sum_{\substack{a\in A\,|\\ b=\sigma(a)\in A}} \mathrm{Tr}(M_bM_{b^*}B_{h(b)} - B_{t(b)}M_{b^*}M_b) \\ &\quad - \sum_{\substack{a\in A\,|\\ c=\sigma(a)^*\in A}} \mathrm{Tr}(M_cM_{c^*}B_{h(b)} - B_{t(b)}M_{c^*}M_c) \\ &= s(\sigma)(\mu(M)\cdot B)\end{aligned}$$

where in the second equality we use the fact that $\sigma(a^*) = \sigma(a)^*$. □

Corollary 5.6. *Let $\sigma \in \mathrm{Aut}_*(\overline{Q})$ be an involution and suppose that the dimension vector d and stability parameter θ are σ-compatible and that $\eta \in \mathfrak{g}^*_{Q,d}$ is coadjoint* *fixed and satisfies* $\sigma(\eta) = s(\sigma)\eta$. *Then* σ *preserves* $\mu^{-1}(\eta)$ *and there is an induced automorphism* $\sigma : \mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d} \longrightarrow \mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$.

We can now describe the geometry of the fixed locus of an involution.

Proposition 5.7. *Let* $\sigma \in \mathrm{Aut}_*(\overline{Q})$ *be an involution. If* σ *is* $\overline{Q}$*-anti-symplectic, then* $\mathrm{Rep}^{\sigma}_{\overline{Q},d} \cong \mathrm{Rep}_{Q,d} \neq \emptyset$ *and this fixed locus is a Lagrangian subvariety of* $\mathrm{Rep}_{\overline{Q},d}$. *If* σ *is* $\overline{Q}$*-symplectic, then* $\mathrm{Rep}^{\sigma}_{\overline{Q},d}$ *is a symplectic subvariety of* $\mathrm{Rep}_{\overline{Q},d}$. *When the symplectic reduction* $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ *is a smooth algebraic variety, the fixed locus of the induced involution is therefore Lagrangian if* σ *is* $\overline{Q}$*-anti-symplectic and symplectic if* σ *is* $\overline{Q}$*-symplectic.*

Proof. If σ is a $\overline{Q}$-anti-symplectic involution, no arrows are fixed by σ and $\overline{A} = A \sqcup A^* = A \sqcup \sigma(A)$. Hence $\mathrm{Rep}^{\sigma}_{\overline{Q},d} \cong \mathrm{Rep}_{Q,d}$ (in particular, this has half the dimension of $\mathrm{Rep}_{\overline{Q},d}$). The remaining statements follow from general properties of anti-symplectic and symplectic involutions of a non-singular symplectic variety (X,ω) over a field of characteristic $\neq 2$: if $\sigma : X \longrightarrow X$ is of order 2, the tangent space at x is $T_xX = \ker(T_x\sigma - \mathrm{Id}) \oplus \ker(T_x\sigma + \mathrm{Id})$; these two subspaces are Lagrangian if $\sigma^*\omega = -\omega$ and each other's symplectic complement if $\sigma^*\omega = \omega$. Note that one of these relations indeed holds here, in view of Proposition 5.5 □

5.2. **The hyperkähler case.** Over the complex numbers, the algebraic symplectic reduction has a hyperkähler structure, as it can be interpreted as a hyperkähler reduction via the Kempf–Ness theorem. In fact, we can generalise the above situation as follows. Let $X = \mathbb{A}^n_{\mathbb{C}}$ be a complex affine space with a linear action of a complex reductive group G. We can assume without loss of generality that the maximal compact subgroup U of G acts unitarily (by rechoosing coordinates on X if necessary). The standard Hermitian form $H : X \times X \longrightarrow \mathbb{C}$ given by $(z,w) \longmapsto z^t\overline{w}$, where we consider w and z as row vectors, is then U-invariant. In particular, X is a Kähler manifold with complex structure given by multiplication by $i \in \mathbb{C}$, metric $g = \mathrm{Re}H$ and symplectic form $\omega_X = -\mathrm{Im}H$. The action of U on X is symplectic for ω_X with moment map $\mu_X : V \longrightarrow \mathfrak{u}^*$ given by $\mu_{\mathbb{R}}(z) \cdot B = \frac{i}{2}H(Bz,z)$ where $z \in X$ and $B \in \mathfrak{u}^*$. The cotangent bundle is hyperkähler, as we can identify $T^*X \cong X \times X^* \cong \mathbb{H}^n$ by $(z,\alpha) \longmapsto (z - \alpha j)$ and inherit the hyperkähler structure from the quaternionic vector space $\mathbb{H}^n$. Let I,J and K denote the complex structures on T^*X obtained from the complex structures on $\mathbb{H}^n$ given by left multiplication by i, j, k. We consider the associated symplectic forms

$$\omega_I(-,-) = g(I-,-), \quad \omega_J(-,-) = g(J-,-) \quad \text{and} \quad \omega_K(-,-) = g(K-,-).$$

We often write $\omega_{\mathbb{R}} = \omega_I$ and $\omega_{\mathbb{C}} = \omega_J + i\omega_K$, which is the Liouville algebraic symplectic form. This linear G-action on X lifts to a linear action of G (and U) on T^*X; moreover, as U acts unitarily on X, the U-action is symplectic with respect to $\omega_{\mathbb{R}}$ and the G-action is symplectic with respect to $\omega_{\mathbb{C}}$. The associated moment maps $\mu_{\mathbb{R}} : T^*X \longrightarrow \mathfrak{u}^*$ and $\mu_{\mathbb{C}} : T^*X \longrightarrow \mathfrak{g}^*$ are given by $\mu_{\mathbb{R}}(z,\alpha) \cdot B = (\mu_X(z) - \mu_X(\alpha)) \cdot B$ and $\mu_{\mathbb{C}}(z,\alpha) \cdot A = \alpha(A^{\#}_z)$, where $A \in \mathfrak{g}$ and $(z,\alpha) \in T^*X$ and $A^{\#}_z$ denotes the infinitesimal action of A on z. We often write $\mu_{HK} := (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$, which is a hyperkähler moment map for the U-action on T^*X. Let $\chi \in \mathfrak{u}$ and $\eta \in \mathfrak{g}$ be coadjoint fixed, then U acts on the level set $\mu_{HK}^{-1}(\chi,\eta) := \mu_{\mathbb{R}}^{-1}(\chi) \cap \mu_{\mathbb{C}}^{-1}(\eta)$ and the hyperkähler reduction is the topological quotient $\mu_{HK}^{-1}(\chi,\eta)/U$, which inherits an orbifold hyperkähler structure if U acts with finite stabilisers on $\mu_{HK}^{-1}(\chi,\eta)$. By the Kempf-Ness theorem [18], there is a homeomorphism between the hyperkähler reduction and algebraic symplectic reduction: $\mu_{HK}^{-1}(\chi,\eta)/U \cong \mu_{\mathbb{C}}^{-1}(\eta)/\!/_{\chi}G$, where we consider χ as a character of G by complexifying and exponentiating.

For a quiver Q, we can apply the above picture to $G = \mathbf{G}_{Q,d}$ acting on $X = \mathrm{Rep}_{Q,d}$. Then $U = \mathbf{U}_{Q,d} := \Pi_{v\in V}\mathbf{U}(d_v)$ and we take the Hermitian form on $\mathrm{Rep}_{Q,d}$ given by $H(M,N) = \sum_{a\in A}\mathrm{Tr}(M_a\, {}^t\overline{N}_a)$. The hyperkähler metric g on $\mathrm{Rep}_{\overline{Q},d}$ is given by

$$g(X,Y) = \mathrm{Re}\Big(\sum_{a\in\overline{A}}\mathrm{Tr}(X_a\, {}^t\overline{Y}_a)\Big); \tag{5.3}$$

therefore, $\omega_{\mathbb{R}} = \omega_I$ is given by $\omega_{\mathbb{R}}(X,Y) = \mathrm{Im}\left(\sum_{a\in\overline{A}}\mathrm{Tr}({}^t\overline{X}_aY_a)\right)$ and $\omega_{\mathbb{C}} = \omega_J + i\omega_K$ is the Liouville algebraic symplectic form ω described in (5.1). Moreover, $\mu_{\mathbb{C}} = \mu : \mathrm{Rep}_{\overline{Q},d} \longrightarrow \mathfrak{g}^*_{Q,d}$ is the algebraic moment map given by (5.2) and $\mu_{\mathbb{R}} : \mathrm{Rep}_{\overline{Q},d} \longrightarrow \mathfrak{u}^*_{Q,d}$ can be explicitly described as

$$\mu_{\mathbb{R}}(M)\cdot B = \frac{i}{2}\sum_{a\in\overline{A}}\mathrm{Tr}(B_{h(a)}M_a\, {}^t\overline{M}_a - B_{t(a)}\, {}^t\overline{M}_aM_a).$$

If we use the standard identification $\mathfrak{u}_{Q,d} \cong \mathfrak{u}^*_{Q,d}$, then we can consider the real moment map as a map $\mu_{\mathbb{R}} : \mathrm{Rep}_{\overline{Q},d} \longrightarrow \mathfrak{u}_{Q,d}$ given by $\mu_{\mathbb{R}}(M) = \frac{i}{2}\sum_{a\in\overline{A}}[M_a, {}^t\overline{M}_a]$. If χ_θ-semistability coincides with χ_θ-stability on $\mu^{-1}(\eta)$, then we obtain an orbifold hyperkähler structure on the algebraic variety $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$.

We note that Nakajima quiver varieties can also be constructed in this manner. The effect of complex conjugation on Nakajima quiver varieties is studied in [6], where they show the fixed locus in the Nakajima quiver variety is an ABA-brane; see Corollary 3.10 in *loc. cit.* We will study the geometry of the fixed locus of an automorphism $\sigma \in \mathrm{Aut}_*(\overline{Q})$ that is either $\overline{Q}$-symplectic or $\overline{Q}$-anti-symplectic.

Lemma 5.8. *Let $\sigma \in \mathrm{Aut}_*(\overline{Q})$ and let d be a σ-compatible dimension vector. Then the automorphism σ of $\mathrm{Rep}_{\overline{Q},d}$ has the following properties.*

(1) *σ is holomorphic with respect to I and symplectic with respect to ω_I.*
(2) *If σ is $\overline{Q}$-symplectic, then σ is holomorphic with respect to J and K and symplectic with respect to ω_J and ω_K.*
(3) *If σ is $\overline{Q}$-anti-symplectic, then σ is anti-holomorphic with respect to J and K and anti-symplectic with respect to ω_J and ω_K.*

Proof. As complex conjugation and transposition commute, σ is I-holomorphic. Since the hyperkähler metric g is preserved by σ (*cf.* the explicit form of the metric given in (5.3)), it follows that $\sigma^*\omega_I = \omega_I$. If $M = (M_a, M_{a^*})_{a\in A} \in \mathrm{Rep}_{\overline{Q},d}$, then $J\cdot(M_a, M_{a^*})_{a\in A} = ({}^t\overline{M}_{a^*}, -{}^t\overline{M}_a)_{a\in A}$. Suppose that σ is contravariant; then

$$\sigma(J\cdot(M_a, M_{a^*})) = \sigma({}^t\overline{M}_{a^*}, -{}^t\overline{M}_a) = s(\sigma)(\overline{M}_{\sigma(a^*)}, -\overline{M}_{\sigma(a)})$$

(where $s(\sigma) = \pm 1$ depending on whether σ is $\overline{Q}$-symplectic or $\overline{Q}$-anti-symplectic) and $J\cdot\sigma(M_a, M_{a^*}) = J\cdot({}^tM_{\sigma(a)}, {}^tM_{\sigma(a^*)}) = (-\overline{M}_{\sigma(a^*)}, \overline{M}_{\sigma(a)})$, which gives the compatibility of J and σ. Since $K = IJ$, we can determine the compatibility of K with σ from that of I and J. The final statements about the compatibility of σ with ω_J and ω_K follow from Proposition 5.5. A very similar computation shows the result also holds when σ is covariant. ∎

Assumption 5.9. Given a subgroup $\Sigma \subset \mathrm{Aut}_*(\overline{Q})$, we shall assume that the dimension vector d and the stability parameter θ are Σ-compatible. Let $\eta \in \mathfrak{g}_{Q,d}$ be a coadjoint fixed element such that $\sigma(\eta) = s(\sigma)\eta$ for all $\sigma \in \Sigma$. Then there is an induced Σ-action on $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ by Corollary 5.6. We assume that the θ-semistable locus in $\mu^{-1}(\eta)$ is non-empty and that $\mathbf{G}_{Q,d}$ acts freely on this θ-semistable locus, so that the quotient $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ is smooth and has a natural hyperkähler structure.

Theorem 5.10. *Under Assumption 5.9 for a subgroup $\Sigma \subset \mathrm{Aut}_*(\overline{Q})$ consisting of $\overline{Q}$-symplectic transformations, the Σ-fixed locus in $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ is hyperholomorphic (or, in the language of branes, this fixed locus is a BBB-brane).*

Proof. If σ is $\overline{Q}$-symplectic, then the automorphism σ on $\mathrm{Rep}_{\overline{Q},d}$ is holomorphic with respect to all three complex structures by Lemma 5.8, so the same is true for the induced automorphism on the hyperkähler reduction with respect to its induced complex structures. Therefore, the fixed locus is a holomorphic submanifold with respect to all three complex structures. □

Theorem 5.11. *Under Assumption 5.9 for a $\overline{Q}$-anti-symplectic involution σ, the fixed locus of σ acting on $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ is a BAA-brane.*

Proof. This is a direct consequence of Lemma 5.8 and the construction of the three symplectic structures on the hyperkähler quotient $\mu_{HK}^{-1}(\chi_\theta,\eta)/\mathbf{U}_{Q,d}$. □

If we apply Theorem 5.11 to the anti-symplectic contravariant involution σ that fixes all vertices and on arrows a is given by $\sigma(a) = a^*$, then the σ-fixed locus $(\mu^{-1}(0)/\!/_{\chi_\theta}\mathbf{G}_{Q,d})^\sigma$ has as a connected component the subvariety

$$\mathrm{Rep}^{\sigma}_{\overline{Q},d}/\!/_{\chi_\theta}\mathbf{G}^{\sigma}_{Q,d} = \mathrm{Rep}_{Q,d}/\!/_{\chi_\theta}\mathbf{G}_{Q,d} = \mathcal{M}^{\theta-ss}_{Q,d},$$

which is known to be Lagrangian in $\mu^{-1}(0)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ (cf. [25, Proposition 2.4]).

Let $\tau \in \mathrm{Gal}_{\mathbb{R}}$ denote complex conjugation; then we can consider compositions $\gamma := \tau\circ\sigma$, where $\sigma \in \mathrm{Aut}_*(\overline{Q})$ is a $\overline{Q}$-(anti)-symplectic automorphism and describe the geometry of the associated fixed loci in the language of branes. We focus on the case where σ is also involutive, as this is our main source of applications.

Corollary 5.12. *If Assumption 5.9 holds for an involution σ which commutes with τ, then, when non-empty, the fixed locus of the involution $\sigma\circ\tau$ acting on the hyperkähler manifold $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$ is*

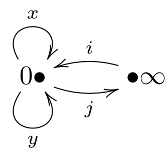
1. *an ABA-brane, if σ is $\overline{Q}$-symplectic (here we allow $\sigma = \mathrm{Id}$);*
2. *an AAB-brane, if σ is $\overline{Q}$-anti-symplectic.*

In particular, we see that all four types of branes (BBB, BAA, ABA and AAB) can be constructed as the fixed locus of an involution. We also note that since τ and all $\overline{Q}$-(anti)-symplectic transformations of $\overline{Q}$ induce isometries of $\mathrm{Rep}_{\overline{Q},d}$ by Lemma 5.8, the fixed loci of the various involutions we have considered are also totally geodesic submanifolds of the hyperkähler quotient $\mu^{-1}(\eta)/\!/_{\chi_\theta}\mathbf{G}_{Q,d}$.

6. Further applications and examples

In this section, we calculate some fixed loci for quiver group actions on the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ and polygon spaces, which can both be realised as quiver moduli spaces.

6.1. **Hilbert scheme of points in the plane.** The Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ of n points in the affine plane over an algebraically closed field k can be realised as a Nakajima quiver variety associated to the Jordan quiver. More precisely, let $\overline{Q}$ be the double of the framed Jordan quiver Q, i.e. $\overline{Q}$ is the following quiver:



where the vertex at infinity is the framing vertex. For $d = (n, 1)$, we have

$$\mathrm{Rep}_{\overline{Q},d} = \mathrm{Mat}_{n\times n} \times \mathrm{Mat}_{n\times n} \times \mathrm{Mat}_{n\times 1} \times \mathrm{Mat}_{1\times n}$$

and for the action of $\mathbf{GL}_n \subset \mathbf{G}_{Q,d}$ (that is, one ignores the group $\mathbb{G}_m$ corresponding to the framing vertex), we have a moment map $\mu : \mathrm{Rep}_{\overline{Q},d} \longrightarrow \mathfrak{gl}_n$ given by

$$\mu(M_x, M_y, M_i, M_j) = [M_x, M_y] + M_i \otimes M_j,$$

where M_i is a column vector and M_j a row vector (viewed as a linear form). The Hilbert scheme is a $\mathbf{GL}_n$-quotient of the level set of the moment map at zero (other level sets give rise to Calogero-Moser spaces). By [10, Section 5.6], the affine GIT quotient of the $\mathbf{GL}_n$-action on $\mu^{-1}(0)$ with respect to the trivial stability parameter is isomorphic to the n-th symmetric power of $\mathbb{A}^2$, i.e. $\mu^{-1}(0)/\!/\mathbf{GL}_n \cong \mathrm{Sym}^n(\mathbb{A}^2)$, and the GIT quotient with respect to the character $\det : \mathbf{GL}_n \longrightarrow \mathbb{G}_m$ (that corresponds to the stability parameter $\theta_0 = -1$) is the Hilbert scheme of n-points on the affine plane, i.e. $\mu^{-1}(0)/\!/_{\det}\mathbf{GL}_n \cong \mathrm{Hilb}^n(\mathbb{A}^2)$, which is a geometric quotient of $\mu^{-1}(0)^{\det -ss} = \mu^{-1}(0)^{\det -s}$, and moreover, the natural morphism from the former to the latter is the Hilbert-Chow morphism, which is an algebraic symplectic resolution of singularities. More precisely, $M := (M_x, M_y, M_i, M_j) \in \mu^{-1}(0)$ is GIT stable for the character det if $[M_x, M_y] = 0$, and $M_j = 0$, and M_i is a cyclic vector for M_x and M_y. In this case, the corresponding point in the Hilbert scheme is given by the ideal $J_M := \{f(x, y) \in k[x, y] : f(M_x, M_y)M_i = 0\} \subset k[x, y]$, which has codimension n, as M_i is cyclic. We note that the ideal J_M is constant on the $\mathbf{GL}_n$-orbit of M.

Note that the present setting is slightly different from the one in Section 5, insofar as one does not consider the action of the full $\mathbf{G}_{Q,d} = \mathbf{GL}_n \times \mathbb{G}_m$ on $\mathrm{Rep}_{\overline{Q},d}$, but only that of the subgroup $\mathbf{GL}_n \subset \mathbf{G}_{Q,d}$ (the automorphism group of the unframed quiver) and consequently, there is no global stabiliser group Δ. In particular, $\mathbf{GL}_n$ acts freely on $\mu^{-1}(0)^{\det -s}$. For our results, this simply means that one should replace Δ with the trivial group and $\mathbf{G}_{Q,d}$ with $\mathbf{GL}_n$.

Let us study the automorphism group of $\overline{Q}$. For reasons of valency, every automorphism of $\overline{Q}$ must fix each vertex. We have that $\mathrm{Aut}(\overline{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the non-trivial automorphisms are $\sigma : x \longmapsto y$ leaving i and j fixed, then $\sigma' : i \longmapsto j$ leaving x and y fixed and finally $\sigma \circ \sigma'$, where σ is covariant, and σ', $\sigma \circ \sigma'$ are contravariant. The dimension vector $d = (n, 1)$ is $\mathrm{Aut}(\overline{Q})$-compatible and every stability parameter is compatible with the covariant automorphisms, but only the trivial notion of stability is compatible with the contravariant involutions (so there is an induced action of the contravariant involutions on $\mathrm{Sym}^n(\mathbb{A}^2)$, but not on $\mathrm{Hilb}^n(\mathbb{A}^2)$). So let us focus on the covariant involution σ; then $\mathrm{Rep}^{\sigma}_{\overline{Q},d} = \{M = (M_x, M_y, M_i, M_j) : M_x = M_y\}$. Suppose that $M \in \mu^{-1}(0)^{\det -s}$ is σ-fixed;

then $M = (M_x, M_x, M_i, 0)$ where M_i is a cyclic vector for M_x. In this case, the corresponding ideal $J_M \subset k[x, y]$ contains the ideal $I = (x - y)$, and so we have closed embeddings $\operatorname{Spec} k[x, y]/J_M \hookrightarrow \operatorname{Spec} k[x, y]/I \hookrightarrow \operatorname{Spec} k[x, y] = \mathbb{A}^2$, where the final morphism can be viewed as the diagonal embedding $D : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$. Hence, the n points corresponding to the ideal J_M all lie on the diagonal line $D(\mathbb{A}^1) \subset \mathbb{A}^2$. This determines a map $(\mu^{-1}(0)^{\det -s})^\sigma \longrightarrow \operatorname{Hilb}^n(D(\mathbb{A}^1)) \cong \operatorname{Sym}^n(\mathbb{A}^1) \cong \mathbb{A}^n$. As σ acts trivially on all vertices, we have that $\mathbf{GL}_n^\sigma = \mathbf{GL}_n$.

Lemma 6.1. *There is an isomorphism* $(\mu^{-1}(0)^{\det -s})^\sigma/\mathbf{GL}_n \cong \operatorname{Hilb}^n(D(\mathbb{A}^1))$.

Proof. Since the map $(\mu^{-1}(0)^{\det -s})^\sigma \longrightarrow \operatorname{Hilb}^n(D(\mathbb{A}^1))$ described above is $\mathbf{GL}_n$-invariant, it descends to a morphism $(\mu^{-1}(0)^{\det -s})^\sigma/\mathbf{GL}_n \longrightarrow \operatorname{Hilb}^n(D(\mathbb{A}^1))$ by the universal property of the GIT quotient. To show this is an isomorphism, we need to describe the inverse map. A codimension n ideal $J \subset k[x, y]/I \cong k[x]$ determines a n-dimensional k-vector space $V = k[x]/J$, and multiplication by x induces an endomorphism $M_x : V \longrightarrow V$ and the inclusion of the multiplicative unit induces a map $M_i : k \longrightarrow V$. Furthermore, the image of M_i under repeated applications of M_x cyclically generates V. Hence, $J = J_M$ for the σ-fixed stable point $M = (\varphi \circ M_x \circ \varphi^{-1}, \varphi \circ M_x \circ \varphi^{-1}, \varphi \circ M_i, 0)$, where $\varphi : V \longrightarrow k^n$ is a chosen isomorphism (we note that different choices of φ correspond to different points in the $\mathbf{GL}_n$-orbit of M). □

By Proposition 4.8 and Lemma 6.1, there is a map $f_\sigma : \operatorname{Hilb}^n(D(\mathbb{A}^1)) \longrightarrow \operatorname{Hilb}^n(\mathbb{A}^2)^\sigma$, which is injective by Proposition 4.11, as σ fixes all vertices of Q. Let us decompose $\operatorname{Hilb}^n(\mathbb{A}^2)^\sigma$ by using the type map. First, we note that there is only one fibre of the type map, as $H^2(\mathbb{Z}/2\mathbb{Z}, \{1\}) = 1$. By Theorem 4.21, the trivial fibre of the type map has a decomposition indexed by $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbf{GL}_n(k))/\{1\}$, where the $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbf{GL}_n(k)$ is trivial. We note that $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbf{GL}_n(k))$ is in bijection with the set of conjugacy classes of $n\times n$-matrices of order 2. The minimum polynomial of such a matrix divides $x^2 - 1$, and so this matrix is diagonalisable with eigenvalues equal to ± 1. Therefore, $H^1(\mathbb{Z}/2\mathbb{Z}, \mathbf{GL}_n(k)) \cong \{u_0, \dots, u_n\}$, where u_r is the diagonal $n \times n$-matrix with -1 appearing r times on the diagonal followed by 1 appearing $n - r$ times. The element $u_0 = I_n$ corresponds to the trivial modifying family, which does not alter the action. For $r > 0$, we have

$$\operatorname{Rep}_{\overline{Q},d}^{\sigma,u_r} = \{M = (M_x, M_y, M_i, M_j) : M_x = M_y,\ u_r M_i = M_i \text{ and } M_j u_r = M_j\},$$

thus, $(M_i)_l = 0 = (M_j)_l$ for $1 \leq l \leq r$. In particular, for $u_n = -I_n$, the intersection of this fixed locus with $\mu^{-1}(0)^{\det -s}$ is trivial, as for M to lie in this fixed locus, we must have $M_i = 0$, which cannot be a cyclic vector. Moreover, $\mathbf{GL}_n^{\sigma,u_r}$ is the centraliser of u_r in $\mathbf{GL}_n$, and so $\mathbf{GL}_n^{\sigma,u_r} \simeq \mathbf{GL}_r \times \mathbf{GL}_{n-r}$. Hence, we have a decomposition into varieties

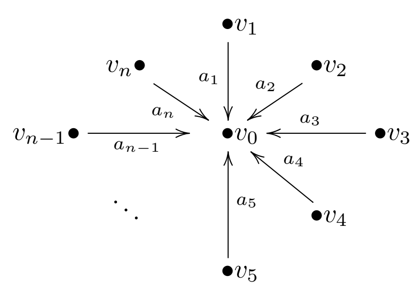
$$\left(\operatorname{Hilb}^n(\mathbb{A}^2)\right)^\sigma \simeq \bigsqcup_{r=0}^{n} (\operatorname{Rep}_{\overline{Q},d}^{\sigma,u_r} \cap \mu^{-1}(0)^{\det -s})/(\mathbf{GL}_r \times \mathbf{GL}_{n-r}).$$

This fixed locus is not a brane (note that σ is neither $\overline{Q}$-symplectic or $\overline{Q}$-anti-symplectic because $A = \{x, i\}$ and $A^* = \{y, j\}$ so $\sigma(A) \nsubseteq A$ and $\sigma(A) \nsubseteq A^*$).

Remark 6.2. One could also consider the Γ-equivariant Hilbert schemes of n-points in the plane, for a finite group $\Gamma \subset \mathrm{SL}_2$. The McKay correspondence associates to such a finite group Γ (up to conjugacy) an affine Dynkin graph of ADE type and the Γ-equivariant Hilbert scheme is a Nakajima quiver variety associated to this affine Dynkin graph. An interesting question, which we do not pursue here, is whether the fixed loci for subgroups of the automorphism group of these quivers have a special representation-theoretic interpretation. For affine Dynkin diagrams of type A, the work of Henderson and Licata [13] shows this to be the case.

6.2. **Moduli of points on $\mathbb{P}^1$ and polygon spaces.** For $n > 1$, fix a tuple of positive integer weights $r = (r_1, \ldots, r_n)$ and an algebraically closed field k. Let $\mathcal{M}_{\mathbb{P}^1}^{r-ss}$ be the moduli space of r-semistable n ordered points on $\mathbb{P}^1$ (over k) modulo the automorphisms of the projective line, where n ordered points $(p_1, \ldots, p_n)$ on $\mathbb{P}^1$ are r-semistable if, for all $p_0 \in \mathbb{P}^1$, $\sum_{i \mid p_i = p_0} r_i \leq \sum_{i \mid p_i \neq p_0} r_i$. This moduli space can be constructed via GIT as a quotient of the SL_2-action on $(\mathbb{P}^1)^n$ with respect to an ample linearisation $\mathcal{L}_r$ associated to the weights. Over the complex numbers, via the Kempf-Ness Theorem, the moduli space $\mathcal{M}_{\mathbb{P}^1}^{r-ss}(n)$ is homeomorphic to the polygon space $\mathcal{M}_{\mathrm{poly}}(n, r)$ consisting of n-gons in the Euclidean space $\mathbb{E}^3$ with lengths given by r modulo orientation-preserving isometries [9, 11, 15].

The moduli space $\mathcal{M}_{\mathbb{P}^1}^{r-ss}(n)$ is isomorphic to a moduli space of representations for the star-shaped quiver Q_n with one central vertex v_0 and n outer vertices $v_0, \ldots, v_n$ and arrows $a_i : v_i \longrightarrow v_0$



with dimension vector $d = (2, 1, \cdots, 1) \in \mathbb{N}^{n+1}$, which we will often suppress from the notation, and stability parameter $\theta_r := (-\sum_{i=1}^n r_i, 2r_1, 2r_2, \cdots, 2r_n)$. Hence, $\mathrm{Rep}_{Q_n,d} \cong (\mathbb{A}^2)^n$ and a closed point $M \in \mathrm{Rep}_{Q_n,d}$ is a tuple $M = (M_1, \cdots, M_n)$ of k-linear maps $M_i : k \longrightarrow k^2$; we note that the injectivity of all of these maps M_i is a necessary condition for semistability for any tuple of positive weights r. In particular, a semistable point M determines n ordered points in $\mathbb{P}^1 \simeq \mathbb{A}^2 /\!/_{\det_{r_i}} \mathbb{G}_m$ by taking the lines in k^2 given by the images of the injective linear maps M_i. Since $\mathbf{G}_{Q_n,d} = \mathbf{GL}_2 \times \mathbb{G}_m^n$, it follows that

$$\mathcal{M}_{Q_n,d}^{\theta_r-ss} := (\mathbb{A}^2)^n /\!/_{\chi_{\theta_r}} (\mathbf{GL}_2 \times \mathbb{G}_m^n) \cong (\mathbb{P}^1)^n /\!/_{\mathcal{L}_r} \mathbf{GL}_2 =: \mathcal{M}_{\mathbb{P}^1}^{r-ss}(n).$$

Every automorphism of Q_n must fix v_0, so $\mathrm{Aut}(Q_n) = \mathrm{Aut}^+(Q_n) \cong S_n$. The dimension vector d is $\mathrm{Aut}(Q_n)$-compatible, but the stability parameter θ_r is only $\mathrm{Aut}(Q_n)$-compatible in the equilateral case, where all weights r_i coincide. By restricting our attention to subgroups $\Sigma \subset \mathrm{Aut}(Q_n)$, there are more weight vectors which are Σ-compatible. For instance, given any subset $I \subset \{1, \cdots, n\}$ of size $1 \leq m = |I| \leq n$, one can consider two remarkable subgroups of $\mathrm{Aut}(Q_n)$: a cyclic group $\Sigma_I \cong \mathbb{Z}/m\mathbb{Z}$ and a symmetric group $\Sigma_I' \cong S_m$, permuting the vertices indexed by I. For θ_r to be Σ_I-compatible (or, equivalently, Σ_I' -compatible), we need the weight vector r to satisfy $r_i = r_j$ for all i, j in I. The quotient quivers for both Σ_I and Σ_I' agree: $Q_n/\Sigma_I = Q_n/\Sigma_I' \simeq Q_{n-m+1}$. If, for notational simplicity, we suppose $I = \{1, \cdots, m\}$, then the induced stability parameter $\tilde{\theta_r} = (-\sum_{i=1}^n r_i, 2mr_1, 2r_{m+1}, \cdots, 2r_n)$ on Q_{n-m+1} is the stability parameter associated to the weight vector $r_I = (mr_1, r_{m+1}, \cdots, r_n)$. By Proposition 4.8, there are morphisms $f_{\Sigma_I^{(\prime)}} : \mathcal{M}_{Q_{n-m+1}}^{\theta_{r_I}-ss} = \mathrm{Rep}_{Q_n,d}^{\Sigma_I^{(\prime)}} /\!/_{\chi_{\theta_r}} \mathbb{G}_{Q_n}^{\Sigma_I^{(\prime)}} \longrightarrow (\mathcal{M}_{Q_n}^{\theta_r-ss})^{\Sigma_I^{(\prime)}}$, whose restriction to the regularly stable locus is injective by Proposition 4.11, as the vertex v_0 is fixed by every automorphism of Q_n. Over the complex numbers, $f^{rs}_{\Sigma_I^{(\prime)}}$ can be identified with the inclusion $\mathcal{M}_{\mathrm{poly}}(n-m+1, r_I) \hookrightarrow \mathcal{M}_{\mathrm{poly}}(n, r)$. The fibres of the type map can be decomposed using Theorem 4.21.

For example, consider $\sigma = (12)$ and fix a weight vector r with $r_1 = r_2$. The type map has only one fibre, as $H^2(<\sigma>, \Delta(k)) = 1$. Moreover, by Theorem 4.21, the components of $(\mathcal{M}_{Q_n}^{\theta_r-s})^\sigma$ are indexed by $H^1(\Sigma, \mathbf{G}_{Q_n,d}(k))/H^1(\Sigma, \Delta(k))$, where $\Sigma :=<\sigma>$. Note first that $H^1(\Sigma, \Delta(k)) \simeq \{\pm 1\}$. Let $\beta : \Sigma \longrightarrow \mathbf{G}_{Q_n,d}(k)$ be a normalised 1-cocycle; then β is given by $\beta(\sigma) = (u_0, u_1, \cdots, u_n) \in \mathbf{G}_{Q_n,d}(k)$ such that $u_0^2 = I_2$, $u_1 u_2 = 1$ and $u_i^2 = 1$ for $i = 3, \cdots n$. Two such cocycles β and β' are cohomologous if there exists $g \in \mathbf{G}_{Q_n,d}(k)$ such that $g\beta(\sigma)\Psi_\sigma(g^{-1}) = \beta'(\sigma)$; that is, if and only if $g_0 u_0 g_0^{-1} = u_0'$ and $g_1 u_1 g_2^{-1} = u_1'$ and $u_i = u_i'$ for $i = 3, \cdots, n$. Thus, to describe the cohomology class, we can assume that u_0 is in Jordan normal form and $u_1 = u_2 = 1$. Since $u_0^2 = I_2$, we can assume that $u_0 \in \{\pm I_2, A := \mathrm{diag}(-1, 1)\}$. Hence, there is a bijection between $H^1(\Sigma, \mathbf{G}_{Q_n,d}(k))/H^1(\Sigma, \Delta(k))$ and the set

$$\mathcal{B} := \{(u_0, \ldots, u_n) : u_0 \in \{I_2, A\},\ u_1 = u_2 = 1,\ u_i = \pm 1 \text{ for } i = 3, \cdots, n\},$$

which gives 2^{n-1} possible components of the fixed locus, although it turns out that some may be empty, as we shall now see.

Case 1. Suppose $u_0 = I_2$. If there exists $3 \leq i \leq n$ with $u_i = -1$, then for $M = (M_1, \ldots, M_n) \in \mathrm{Rep}_{Q_n,d}$ to lie in the fixed locus for the modified action given by Φ^u, we need $M_i = 0$; in particular, the intersection of this fixed locus with the semistable set is always empty and so such a modifying family u gives an empty contribution. The only non-empty contribution with $u_0 = I_2$ is the trivial element $u = 1$ in $\mathbf{G}_{Q_n,d}$, whose modified action is the original action Φ.

Case 2. Suppose $u_0 = A$. Then $(M_1, \cdots, M_n) \in \mathrm{Rep}_{Q_n,d}$ is fixed for the modified action defined by u, if $M_2 = AM_1$ and, for $3 \leq i \leq n$, the image of M_i is contained in the span of $(0, 1) \in k^2$ (resp. $(1, 0) \in k^2$) if $u_i = 1$ (resp. $u_i = -1$). Let $M \in \mathrm{Rep}_{Q_n,d}^{\Sigma,u}$ be θ_r-semistable; then each M_i corresponds to a point $p_i \in \mathbb{P}^1$, and, for $3 \leq i \leq n$, we have $p_i = [0 : 1]$ if $u_i = 1$ and $p_i = [1 : 0]$ if $u_i = -1$. As $M_2 = AM_1$, we have either $p_1 = p_2 = [0 : 1]$ or $p_1 = [1 : a]$ and $p_2 = [1 : -a]$ for $a \in k$. Moreover, $\mathbf{G}_{Q,d}^{\sigma,u} \cong \mathbb{G}_m^{n+2}$ and the image of $f_{\sigma,u}$ in $\mathcal{M}_{\mathbb{P}^1}^{r-ss}$ is contained in the locus where for $i \geq 3$, we have $p_i = [0 : 1]$ (resp. $[1 : 0]$) if $u_i = 1$ (resp. -1). In terms of polygons, if we identify $[1 : 0]$ and $[0 : 1]$ with $(0, 0, \pm 1) \in \mathbb{R}^3$, then these configurations are such that all but two arrows point in the z-direction (either up or down depending on the sign of u_i).

Remark 6.3. By considering the doubled quiver $\overline{Q_n}$, one can construct hyperkähler analogues of the polygon space $\mathcal{M}_{\mathrm{poly}}(n, r)$, known as hyperpolygon spaces $\mathcal{H}_{\mathrm{poly}}(n, r)$ [20]. In this case, we have $\mathrm{Aut}(\overline{Q_n}) \cong S_n \times \mathbb{Z}/2\mathbb{Z}$, where $\mathrm{Aut}^+(\overline{Q_n}) = S_n$ and we can consider the contravariant involution σ which fixed all vertices and sends a_i to a_i^* as a generator of $\mathbb{Z}/2\mathbb{Z}$. All covariant (resp. contravariant) automorphisms are $\overline{Q_n}$-symplectic (resp. $\overline{Q_n}$-anti-symplectic). In particular, our results on branes in §5 imply that for any permutation $\gamma \in S_n$ of order 2, we have that

(1) $\mathcal{H}_{\mathrm{poly}}(n,r)^{\gamma}$ is a BBB-brane,
(2) $\mathcal{H}_{\mathrm{poly}}(n,r)^{\sigma\circ\gamma}$ is a BAA-brane,
(3) $\mathcal{H}_{\mathrm{poly}}(n,r)^{\gamma\circ\tau}$ is a ABA-brane,
(4) $\mathcal{H}_{\mathrm{poly}}(n,r)^{\sigma\circ\gamma\circ\tau}$ is a AAB-brane,

where τ denotes complex conjugation.

References

[1] D. Baraglia and L. P. Schaposnik. Real structures on moduli spaces of Higgs bundles. arXiv:1309.1195, 2013.
[2] D. Baraglia and L. P. Schaposnik. Higgs bundles and (A,B,A)-branes. *Commun. Math. Phys.*, 331:1271–1300, 2014.
[3] I. Biswas and O. García-Prada. Anti-holomorphic involutions of the moduli spaces of Higgs bundles. *J. Éc. polytech. Math.*, 2:35–54, 2015.
[4] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups*. Cambridge University Press, second edition, 2010.
[5] H. Derksen and J. Weyman. Generalized quivers associated to reductive groups. *Colloq. Math.*, 94(2):151–173, 2002.
[6] E. Franco, M. Jardim, and S. Marchesi. Branes in the moduli space of framed instantons. arXiv: 1504.05883, 2015.
[7] O. García-Parda and S. Ramanan. Involutions and higher order automorphisms of Higgs bundle moduli spaces. arXiv:1605.05143, 2016.
[8] O. García-Prada and G. Wilkin. Action of the mapping class group on character varieties and Higgs bundles. arXiv:1612.02508, 2016.
[9] I. M. Gel′fand and R. D. MacPherson. Geometry in Grassmannians and a generalization of the dilogarithm. *Adv. in Math.*, 44(3):279–312, 1982.
[10] V. Ginzburg. Lectures on Nakajima's quiver varieties. arXiv: 0905.0686, 2009.
[11] J.-C. Hausmann and A. Knutson. Polygon spaces and Grassmannians. *Enseign. Math. (2)*, 43(1-2):173–198, 1997.
[12] S. Heller and L. Schaposnik. Branes through finite group actions. arXiv:1611.00391, 2016.
[13] A. Henderson and A. Licata. Diagram automorphisms of quiver varieties. *Adv. Math.*, 267:225–276, 2014.
[14] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.
[15] Y. Kamiyama and M. Tezuka. Symplectic volume of the moduli space of spatial polygons. *J. Math. Kyoto Univ.*, 39(3):557–575, 1999.
[16] A. Kapustin and E. Witten. Electric-magnetic duality and the geometric langlands program. *Commun. Number Theory Phys.*, 1(1):1–236, 2007.
[17] G. Kempf. Instability in invariant theory. *Ann. of Math.*, 108(2):299–316, 1978.
[18] G. Kempf and L. Ness. The length of vectors in representation spaces. In *Algebraic Geometry*, volume 732 of *Lecture Notes in Mathematics*, pages 233–243. Springer Berlin / Heidelberg, 1979.
[19] A. D. King. Moduli of representations of finite dimensional algebras. *Quart. J. Math.*, 45:515–530, 1994.
[20] H. Konno. Cohomology rings of toric hyperkähler manifolds. *Internat. J. Math.*, 11(8):1001–1026, 2000.
[21] S. G. Langton. Valuative criteria for families of vector bundles on algebraic varieties. *Ann. of Math. (2)*, 101:88–110, 1975.
[22] J. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.*, 5:121–130, 1974.
[23] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. Springer, third edition, 1993.
[24] P. E. Newstead. *Introduction to moduli problems and orbit spaces*, volume 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[25] N. Proudfoot. *Hyperkahler analogues of Kahler quotients*. PhD thesis, U.C. Berkeley, 2004.
[26] S. Ramanan. Orthogonal and spin bundles over hyperelliptic curves. *Proc. Indian Acad. Sci. Math. Sci.*, 90(2):151–166, 1981.
[27] M. Reineke. Moduli of representations of quivers. In *Trends in representation theory of algebras and related topics*, EMS Ser. Congr. Rep., pages 589–637. Eur. Math. Soc., Zürich, 2008.
[28] A. Robert. Automorphism groups of transcendental field extensions. *J. Algebra*, 16:252–270, 1970.
[29] F. Schaffhauser. Real points of coarse moduli schemes of vector bundles on a real algebraic curve. *J. Symplectic Geom.*, 10(4):503–534, 2012.
[30] F. Schaffhauser. On the Narasimhan-Seshadri correspondence for Real and Quaternionic vector bundles. *J. Differential Geom.*, To appear, 2015. arXiv:1509.02052.
[31] F. Schaffhauser. Finite group actions on moduli spaces of vector bundles. arXiv:1608.03977, 2016.
[32] J. P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.
[33] C. S. Seshadri. Space of unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 85:303–336, 1967.
[34] C. S. Seshadri. Geometric reductivity over arbitrary base. *Advances in Math.*, 26(3):225–274, 1977.
[35] M. B. Young. Self-dual quiver moduli and orientifold Donaldson-Thomas invariants. *Commun. Number Theory Phys.*, 9(3):437–475, 2015.

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