SEMINAR ON FURTHER TOPICS IN GIT

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Our seminar for the summer semester 2016 takes place on Tuesday late afternoons, 18:00 - 19:30 in Arnimallee 3, SR 119.

Introduction

In this seminar, we will build on the foundations studied in the Algebra III lectures on moduli problems and geometric invariant theory. We will first explore links with toric geometry and symplectic geometry, and then study variation of GIT, which describes how the GIT quotient changes as we vary the linearisation.

The main goals of this seminar are as follows.

(1) To explore the relationship between toric geometry and GIT. More precisely, we will explain how any projective normal toric variety can be constructed as a GIT quotient of an affine space, where the action is linearised by a character; this is a theorem of Cox [2] which is described in [3, 4].

(2) To give an introduction to symplectic geometry and symplectic quotients [5]. Then we will prove the Kempf-Ness Theorem which gives a homeomorphism between a projective GIT quotient over the complex numbers and a symplectic quotient [10]; see also [11] §8.

(3) To study how the GIT quotient depends on the choice of linearisation of the action, known as variation of GIT. We will describe a wall and chamber structure on the space of linearisations of a reductive group action such that the quotient only changes as one crosses a wall. Furthermore, we will explain the explicit birational transformations between the GIT quotients produced by wall-crossings [6, 13].

In fact, there are many connections between these different goals: variation of GIT was inspired by similar results on variations of symplectic quotients, and variations of GIT quotients of toric varieties can be described in the language of toric geometry.

Description of the talks

Talk 1: An overview of projective normal toric varieties. We start by giving the abstract definition of a toric variety. We then fix a lattice N and explain how to construct affine toric varieties X_σ from cones $\sigma \subset N_\mathbb{R}$ and normal toric varieties X_Σ from fans Σ. Our central focus is projective normal toric varieties: we describe the orbit-cone correspondence and, in particular, describe toric divisors D_ρ which correspond to rays $\rho \in \Sigma(1)$ in the fan Σ. We prove an explicit description of their divisor class group as

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0, \tag{1}$$

where $M = N^\vee$ is the dual lattice and $\mathrm{Cl}(X_\Sigma)$ denotes the divisor class group; see [7] §3. The main examples to cover are the toric descriptions of (weighted) projective spaces and Hirzebruch surfaces in terms of their fans.

References. The above topics are covered in [7] §1-3 and [3] §1-4.

Talk 2: Toric geometric invariant theory. The aim of this talk is to prove that every projective normal toric variety X_Σ is constructed as a GIT quotient of a diagonalisable group acting on an affine space which is linearised by a character. In the first part of this talk, we explain how to construct GIT quotients of affine spaces with respect to a character following [4] §2. In the second part of the talk, we give the proof of the above result. Let us outline the argument: the starting point is the short exact sequence (1) to which we apply $\mathrm{Hom}(-, k^*)$ to obtain a short exact sequence of groups. We let $G := \mathrm{Hom}(\mathrm{Cl}(X_\Sigma), k^*)$ denote the subgroup of

$(\mathbb{G}_m)^{|\Sigma(1)|}$ occurring in this short exact sequence; then G is a diagonalisable group which acts on $\mathbb{A}^{\Sigma(1)}$ via its embedding into the torus. The character group of G is naturally identified with the class group of X_Σ and we say a character is ample if the corresponding divisor is ample. For an ample character χ, we prove that there is an isomorphism

$$X_\Sigma \cong \mathbb{A}^{\Sigma(1)} /\!/_\chi G.$$

In the proof, we introduce the total coordinate ring (or Cox ring) of X_Σ, whose spectrum is the affine space $\mathbb{A}^{\Sigma(1)}$, and prove the result using the cover X_σ of X_Σ by affine toric varieties. An easy example of this construction is $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1}$ by $G := \mathbb{G}_m$ with respect to an ample character. Using this theorem, we give the construction of Hirzebruch surfaces as quotients of $\mathbb{A}^4$ by $G \cong (\mathbb{G}_m)^2$ with respect to an ample character. In these example, we also study how the quotient varies as we vary the character outside the ample cone.
References. The original reference is [2]; however, the surveys given in [3] §18 and [4] §2-3 are highly recommended references.

Talk 3: An introduction to symplectic geometry. We give the definition of a symplectic manifold; the main examples are symplectic vector spaces, the cotangent bundle of a manifold with its Liouville symplectic form, a coadjoint orbit of a connected compact real Lie group and complex projective space $\mathbb{P}^n_{\mathbb{C}}$ with the Fubini–Study form. For a symplectic vector space, we introduce the notion of (co)isotropic and Lagrangian subspaces and directly prove that any symplectic vector space has even real dimension. Consequently, any symplectic manifold has even dimension over $\mathbb{R}$. We introduce two different types of morphisms in symplectic geometry: symplectomorphisms (which are diffeomorphisms which respect the symplectic form) and Lagrangian correspondences (which are defined in terms of Lagrangian submanifolds of the product, which essentially play the role of the graph of the morphism). Finally, we define Hamiltonian and symplectic vector fields on a symplectic manifold.
References. [5] §1-3 (and §18).

Talk 4: Group quotients in symplectic geometry. In this talk, we describe the construction of symplectic quotients of actions of compact groups in symplectic geometry, which is known as symplectic reduction and is performed by taking a topological quotient of a level set of a moment map for the action. To start, for a compact group K acting on a symplectic manifold (X, ω), we define the infinitesimal action and give the notion of a symplectic action. We explain that as symplectic manifolds are even dimensional, for dimension reasons alone the topological quotient, even if it gives a smooth manifold, may not be symplectic. Instead we produce a symplectic quotient with expected dimension equal to $\dim X - 2\dim K$, by using a moment map $\mu : X \to \mathfrak{k}^*$, where $\mathfrak{k}$ denotes the Lie algebra of K. We introduce the moment map by first defining the co-moment map, which we can intuitively think of as a lift of the infinitesimal action. Then using the moment map, we define the symplectic reduction. Finally, we prove the Marsden-Weinstein-Meyer Theorem, which states that if the group acts freely on this level set, then the reduction is a symplectic manifold. We will use without proof the slice theorem for manifolds which states if a group acts freely and properly on a manifold, then the topological quotient has a unique structure of a manifold for which the quotient map is a submersion.

If time permits, we will explain the idea of symplectic implosion, which can be thought of an abelianisation process: for a symplectic action of K on (X, ω), one associates a symplectic implosion X_{impl} with a symplectic action of a maximal torus T of K, such that all symplectic reductions of the K-action on X can be constructed as symplectic reductions of the T-action on X_{impl}.
References. [5] §21-24; see also [11] §8.1; for symplectic implosion see [8]

Talk 5: The Kempf-Ness theorem. We first explain that there is a 1-1 correspondence between complex reductive groups and compact Lie groups, given by associating to a complex reductive group G its maximal compact subgroup K. In this talk, we prove the Kempf-Ness theorem which states that over the complex numbers, the GIT quotient of a reductive group, G, acting linearly on a smooth projective variety $X \subset \mathbb{P}^n$ is homeomorphic to the symplectic reduction of the action of the maximal compact subgroup K acting on the symplectic manifold X (where the symplectic structure is inherited from the Fubini–Study form on $\mathbb{P}^n$). The main step is to prove that a G-orbit meets the zero level set of the moment map $\mu^{-1}(0)$ if and only if the orbit is polystable. Moreover, in this case, the G-orbit meets $\mu^{-1}(0)$ in a unique K-orbit. From the first statement, it follows that $\mu^{-1}(0) \subset X^{ss}$ and so this inclusion induces a continuous map

$$\mu^{-1}(0)/K \to X/\!\!/G,$$

which is a bijection by what we have said above. Since any continuous bijection from a compact space to a Hausdorff space is a homeomorphism, we obtain the desired result.
References. The result is given by Kempf-Ness [10]; for excellent surveys, see [11] §8.2 and [14].

Talk 6: The Bialynicki-Birula decomposition. Given a $\mathbb{G}_m$-action on a smooth projective variety X, we prove that there are two decompositions of X into locally closed smooth subvarieties $X_i^{\pm}$ which are indexed by the connected components X_i of the fixed point locus $X^{\mathbb{G}_m}$. We explain that in this situation the fixed locus is smooth. Let us explain the construction of this decomposition: X_i^+ (resp. X_i^-) is defined to be the set of points $x \in X$ whose limit under the action of $t \in \mathbb{G}_m$ as $t \to 0$ (resp $t \to \infty$) lies in the fixed point set X_i. We prove that the natural retractions $X_i^{\pm} \to X_i$ are locally trivial affine fibrations (with an action by $\mathbb{G}_m$) and we can explicitly describe the weights of the $\mathbb{G}_m$-action on the normal bundle to $X_i \subset X$ and $X_i \subset X_i^{\pm}$. As an example, we can consider actions of $\mathbb{G}_m$ on $\mathbb{P}^n$ and write down the associated decompositions.

If time permits, we explain that this result holds under weaker assumptions: if $\mathbb{G}_m$ acts on a smooth quasi-projective variety X and $X^{\mathbb{G}_m}$ is projective and the limit as $t \to 0$ exists for all points in X (or the limit as $t \to \infty$ exists for all X), then we can construct a Bialynicki-Birula decomposition.
References. The original reference [1] is still the best.

Talk 7: Variation of GIT for the multiplicative group acting on an affine varieties. In this talk, we explain the results of [13] §1 on the variation of GIT quotients for a linear action of the multiplicative group $\mathbb{G}_m$ on an affine variety X. We consider linearisations of the action given by characters of $\mathbb{G}_m$ in $X_*(\mathbb{G}_m)_{\mathbb{R}} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$ and, as the GIT quotient by χ and χ^n are isomorphic for $n \in \mathbb{N}$, it suffices to consider the linearisations (denoted $-, 0, +$) given by the characters corresponding to the elements $-1, 0, 1 \in \mathbb{Z}$. We let $X^{\pm}$ be the set of $x \in X$ such that the limit of $t \cdot x$ as $t \to 0$ (resp $t \to \infty$) exists in X and show $X^{ss}(\pm) = X - X^{\mp}$.

The main results are as follows. First, we prove that there are birational morphisms $X/\!\!/_{\pm}\mathbb{G}_m \to X/\!\!/_0\mathbb{G}_m$ and describe the locus on which these morphisms are isomorphisms. We prove that the birational transformation $X/\!\!/_+\mathbb{G}_m \dashrightarrow X/\!\!/_-\mathbb{G}_m$ is a flip, if $X^{\pm} \subset X$ has codimension ≥ 2. Then we describe this transformation explicitly as a blow up followed by a blow down (see [13] Theorem 1.9). Finally, if X is smooth, we prove [13] Corollary 1.13 that over $X^0 := X^{\mathbb{G}_m}$, the morphisms $X/\!\!/_{\pm}\mathbb{G}_m \to X/\!\!/_0\mathbb{G}_m$ are locally trivial fibrations with fibres given by weighted projective spaces, where the weights are weights appearing in the normal bundles for the Bialynicki-Birula decomposition associated to the action.

If there is time, the statement and proof of Theorem 1.15 should be given and it would be nice to cover the example of the Atiyah flip; see [13] Example 1.16.
References. The main references for VGIT are [6, 13]; we follow [13] §1 in this talk.

Talk 8: Toric quotients and flips. First we describe the toric variety associated to a polyhedron (which can be thought of as dual to the approach using fans). Then we study quotients of such toric varieties X by subtori S of the torus T_X for the toric variety following [12] §1-3.

More precisely, we show that the linearisations in the space $X_*(S)_{\mathbb{R}}$ of characters of S which give rise to non-empty quotients are parametrized by a polyhedron, which is the projection onto a subspace of the polyhedron defining the toric variety. Furthermore, we show that this polyhedron is partitioned by walls into polyhedral chambers inside which the quotient is essentially constant. Finally, we show that moving between adjacent chambers (a process known as

wall-crossing) induces a birational map between the associated GIT quotients which, in good cases, is a flip.
References. The main reference is [12].

Talk 9: The space of linearisations. In this talk, we explore the structure of a parameter space of linearisations of a given action of a reductive group G on a quasi-projective variety X. First, we will introduce the notion of G-algebraic equivalence on $\mathrm{Pic}^G(X)$ analogous to algebraic equivalence on $\mathrm{Pic}(X)$ and we will show that two linearisations which are G-algebraic equivalent give the same quotient. Consequently the different quotients are parametrised by the G-Néron-Severi group, denoted NS^G. We define the G-effective cone, $E^G_{\mathbb{Q}}$, inside $\mathrm{NS}^G_{\mathbb{Q}} = \mathrm{NS}^G \otimes \mathbb{Q}$, where E^G consists of equivalence classes of linearisations $\mathcal{L}$ such that $X^{ss}(\mathcal{L}) \neq \varnothing$.

The main result is that there is a wall and chamber structure on $E^G_{\mathbb{Q}}$ where the GIT quotients are the same with respect to linearisations inside a fixed chamber and the chambers are given by connected components of the complements of the walls. Moreover, we prove that there are only finitely many chambers.
References. We follow [13] §2 & 3, but everything (and more) can be found in [6] §3.

Talk 10: Wall-crossing in variation of GIT. For an action of a reductive group G on a projective over affine variety X, the idea of this talk is to generalise some of the results in Talk 7 on wall-crossings. We suppose we have two ample linearisations $\mathcal{L}_\pm$ in adjacent chambers to a wall such that there is a line segment $[-1,+1]$ joining these linearisations which crosses the wall at t_0 and such that if $\mathcal{L}(t) \coloneqq \mathcal{L}_+^t \otimes \mathcal{L}_-^{1-t}$, we have

$$X^{ss}(t) = \begin{cases} X^{ss}(+) & \text{if } t_0 < t \leq 1, \\ X^{ss}(-) & \text{if } -1 \leq t < t_0. \end{cases}$$

Then we consider the triple of linearisations $\mathcal{L}_\pm$ and $\mathcal{L}_0 \coloneqq \mathcal{L}(t_0)$.

The idea is to associate to the G-action on X linearised by $(\mathcal{L}_\pm, \mathcal{L}_0)$ a 'master space' $Z \coloneqq \mathbb{P}(L_+ \oplus L_-)/\!/G$ with an action of torus $T \cong \mathbb{G}_m$ such that $X/\!/_*G \cong Z/\!/_*T$ for $* \in \{0, \pm\}$ and then apply the previous results for actions of the multiplicative group. More generally, given a $n+1$ tuple of linearisations, one can similarly construct a n-simplicial family of fractional linearisations and a master space Z with an action of $T \cong \mathbb{G}_m^n$.

We show that if $X/\!/_\pm G$ are non-empty, then $X/\!/_\pm G \to X/\!/_0 G$ are proper and birational, and if they are both small, then $X/\!/_- G \dashrightarrow X/\!/_+ G$ is a flip with respect to $\mathcal{O}(1) \to X/\!/_+ G$, where $\mathcal{O}(1)$ denotes the ample line bundle on the GIT quotient $X/\!/_+ G$ determined by $\mathcal{L}_+$ on X. We then study the birational geometry of $X/\!/_\pm G \to X/\!/_0 G$ and find that the irreducible component of $X/\!/_+ G \times_{X/\!/_0 G} X/\!/_- G$ dominating $X/\!/_0 G$ is naturally isomorphic to both the blow up of $X/\!/_\pm G$ along $\mathcal{I}_\pm/\!/_\pm G$ and of $X/\!/_0 G$ along $(\mathcal{I}_+ + \mathcal{I}_-)/\!/_0 G$, where $\mathcal{I}_\pm$ are certain ideal sheaves on X. Hence, we can explicitly describe this birational transformation as a blow up followed by a blow down.
References. These are the main results presented in [13] §3.

Talk 11: The (Zariski local) description of the morphism to the wall. In this talk and the following talk our goal is to extend the main result from Talk 7 on $\mathbb{G}_m$-actions to general reductive group actions under certain additional hypothesis. More precisely, using the notation and assumptions of Talk 10, we want to give a result concerning the local triviality of $f_\pm : X/\!/_\pm G \to X/\!/_0 G$. In this talk, we give such a result under strong assumptions and in the next talk, we prove a local triviality result in the étale topology under slightly weaker assumptions.

We assume [13] (4.4): X is smooth at $x \in X^0 \coloneqq X^{ss}(0) - (X^{ss}(+) \cup X^{ss}(-))$ with stabiliser $G_x \cong \mathbb{G}_m$ and orbit $G \cdot x$ closed in $X^{ss}(0)$. Let $v_\pm$ denote the weight of G_x acting on $\mathcal{L}_{\pm,x}$. If (4.4) holds and we additionally assume the strong hypothesis that $(v_+, v_-) = 1$, then we prove that locally over a neighbourhood of the orbit of x the morphisms $f_\pm$ are Zariski locally trivial weighted projective space fibrations; see [13] Theorem 4.7. If there is time, we will also explain Theorem 4.8 of loc. cit.
References. [13] §4

Talk 12: The (étale local) description of the morphism to the wall. The aim of this talk is to weaken the above strong assumption of Talk 11 at the price of only being able to describe the morphisms $f_\pm$ over a neighbourhood of an orbit $G \cdot x$, where $x \in X$ satisfies the assumptions of [13] (4.4), as étale locally trivial weighted space fibrations. More precisely, we will assume that either the characteristic of our field k is zero or is coprime to (v_+, v_-); this is Assumption (5.2) of loc. cit, where we continue to use the notation introduced in Talk 11.

A key technical tool for this talk is Luna's étale slice theorem, which generalises the slice theorem for actions on smooth manifolds. The first part of the talk should give the statement of this theorem and, if there is time, it would be nice to give some ideas of the proof.

The second part of this talk is to give the proof of [13] Theorem 5.6: for $x \in X$ satisfying the assumptions (4.4) and (5.2) of loc. cit., we prove that locally over a neighbourhood of the orbit of x the morphisms $f_\pm$ are étale locally trivial weighted projective space fibrations.

Finally, if there is time, we prove Theorem 4.9 of loc. cit. and explain why these hypothesis are necessary, by means of a counter example; see Example 5.8 of loc. cit.
References. [13] §5

Talk 13: Mori Dream Spaces and GIT. In this talk we study varieties with a good Mori chamber decomposition, so called 'Mori dream spaces'. First, we define a Mori chamber by introducing an equivalence relation on the Neron-Severi group: we say two $\mathbb{Q}$–Cariter divisors, D_1 and D_2, are Mori equivalent if the rational maps $f_{D_j} : X \to \operatorname{Proj}(R(X, D_j))$ have the same Stein factorisation (that is if there is an isomorphism between the images that makes the obvious triangular diagram commute). Then a Mori chamber is the closure of an equivalence class. There are many examples of Mori dream spaces: quasi-smooth projective toric varieties and many GIT quotients to name a few.

The main goal of this talk is to prove that Mori dream spaces are in fact GIT quotients of affine varieties by a torus by generalising the ideas of Talk 2.
References. The main reference is [9]

References

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