AN INTRODUCTION TO SYMPLECTIC GEOMETRY

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1. Overview over the basic notions in symplectic geometry

Definition 1.1. Let V be a m-dimensional $\mathbb{R}$ vector space and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map. We say Ω is ***skew-symmetric*** if $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.

Theorem 1.1. *Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, \dots, u_k, e_1 \dots, e_n, f_1, \dots, f_n$ of V s.t.*

$$\begin{array}{ll} \Omega(u_i, v) = 0, & \forall i \text{ and } \forall v \in V \\ \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j), & \forall i, j \\ \Omega(e_i, f_j) = \delta_{ij}, & \forall i, j. \end{array}$$

In matrix notation with respect to this basis we have

$$\Omega(u, v) = u^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & id \\ 0 & -id & 0 \end{pmatrix} v.$$

In particular we have $\dim(V) = m = k + 2n$.

Proof. The proof is a skew-symmetric version of the Gram-Schmidt process. First let

$$U := \{u \in V | \Omega(u, v) = 0 \text{ for all } v \in V\},$$

and choose a basis $u_1, \dots, u_k$ for U. Now choose a complementary space W s.t. $V = U \oplus W$. Let $e_1 \in W$, then there is a $f_1 \in W$ s.t. $\Omega(e_1, f_1) \neq 0$ and we can assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{aligned} W_1 &= span(e_1, f_1) \\ W_1^\Omega &= \{w \in W | \Omega(w, v) = 0 \text{ for all } v \in W_1\}. \end{aligned}$$

We claim that $W = W_1 \oplus W_1^\Omega$: Suppose that $v = ae_1 + bf_1 \in W_1 \cap W_1^\Omega$. We get that

$$\begin{aligned} 0 &= \Omega(v, e_1) = -b \\ 0 &= \Omega(v, f_1) = a \end{aligned}$$

and therefore $v = 0$. Now let $v \in W$. We have that $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. And therefore

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^\Omega}.$$

This shows the claim. Now we go on: we pick $e_2, f_2 \in W_1^\Omega$ with $\Omega(e_2, f_2) = 1$. Let $W_2 = span(e_2, f_2)$. With the same arguments as above we obtain a decomposition

$$V = U \oplus W_1 \oplus \dots \oplus W_n.$$

Note that the e_i, f_i always come in pairs, because if one of the W_i^Ω would be one dimensional it would actually lie in U. But now we have our desired decomposition. $\square$

Remark. We will call n the rank of the bilinear map Ω.

Definition 1.2. A skew-symmetric bilinear map is called symplectic (or non-degenerate) if $U = \{0\}$ (U as above). The map Ω is then called a linear symplectic structure on V and (V, Ω) is called a symplectic vector space.

Corollary 1.1.1. *As an immediate Corollary of Theorem 1.1 we get that* $\dim(V) = 2n$ *since* $\dim(U) = 0$*. Choosing a basis as in the Theorem we have that*

$$\Omega(u, v) = u^T \begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix} v.$$

The basis $e_1, \dots, e_n, f_1, \dots, f_n$ *is called a symplectic basis.*

Definition 1.3. We have different kind of subspaces of a symplectic vector space. Let W be a subspace of (V, Ω). Denote

$$W^\Omega := \{v \in V | \Omega(v, w) = 0 \ \forall w \in W\}$$

1) A subspace $W \subset V$ is called ***symplectic*** if $W \cap W^\Omega = \emptyset$ if and only if $\Omega_{|W}$ is non-degenerate. For example $span(e_i, f_i)$ is a symplectic subspace.
2) A subspace $W \subset V$ is called ***isotropic*** if $W \subset W^\Omega$ if and only if $\Omega_{|W} \equiv 0$. For example $span(e_1, e_2)$ is isotropic.
3) A subspace is called ***coisotropic*** if $W^\Omega \subset W$ if and only if W^Ω is isotropic. For example codimension 1 subspaces are coisotropic.
4) A subspace is called ***Lagrangian*** if it is isotropic and $\dim(W) = \frac{1}{2}\dim(V)$ if and only if $W^\Omega = W$.

Definition 1.4. Let $(V, \Omega), (V', \Omega')$ be two symplectic vector spaces. A linear map $\varphi : V \to V'$ is called a ***symplectomorphism*** if $\Omega = \varphi^*\Omega'$. If a symplectomorphism exists (V, Ω) and (V', Ω') are called symplectomorphic.

Lemma 1.2. *A linear map* $\varphi : V \to V$ *is a symplectomorphism iff the graph*

$$\Gamma_\varphi := \{(v, \varphi(v)) \in V \times V | v \in V\}$$

is a Lagrangian subspace of $(V \times V, (-\Omega) \times \Omega)$*, where* $(-\Omega) \times \Omega((v, v'), (w, w')) = -\Omega(v, w) + \Omega(v', w')$*.*

Proof. Let φ be a symplectomorphism. Then we have $\Gamma_\varphi \subset \Gamma_\varphi^\Omega$, since

$$\begin{aligned} (-\Omega) \times \Omega((v, \varphi(v)), (w, \varphi(w))) &= -\Omega(v, w) + \Omega(\varphi(v), \varphi(w)) \\ &= -\Omega(v, w) + \Omega(v, w) \\ &= 0. \end{aligned}$$

The dimension of Γ_φ is equal to the dimension of V, which finishes the first direction.

Now let Γ_φ be a Lagrangian subspace it follows that

$$0 = -\Omega(v, w) + \Omega(\varphi(v), \varphi(w)),$$

for all $v, w \in V$. This shows that φ is an isomorphism and $\Omega = \varphi^*\Omega$. $\square$

We now turn to manifolds. Let M be a manifold and ω be a differential 2-form, i.e. for $p \in M$ we have that $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is a skew-symmetric bilinear map and w_p varies smoothly in p. We say ω is closed if it satisfies $d\omega = 0$, i.e. ω is a cycle in the de Rham-complex.

Definition 1.5. The 2-form ω is symplectic if ω is closed and ω_p is a symplectic bilinear map on T_pM for all $p \in M$. If ω is symplectic we obtain

$$\dim(M) = \dim(T_pM) = 2n.$$

A ***symplectic manifold*** is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Example 1.1.

- The prototype for a symplectic manifold is $\mathbb{R}^{2n}$ with linear coordinates $x_1, \dots, x_n$, $y_1, \dots, y_n$. The form
$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$
is symplectic and the set
$$\left\{ \frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_n}(p), \frac{\partial}{\partial y_1}(p), \dots, \frac{\partial}{\partial y_n}(p) \right\}$$
is a symplectic basis of T_pM.
- In a similar spirit we have that $\mathbb{C}^n$ is a symplectic manifold with linear coordinates $z_1, \dots, z_n$ and symplectic form
$$\omega = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k.$$
- Let $M = S^2$ regarded as the set of unit vectors in $\mathbb{R}^3$. We have a form
$$\omega(u, v) = \langle p, u \times v \rangle$$
for $u, v \in T_pS^2 = \{p\}^\perp$. This form is closed since it is of top degree. It is non-degenerate since for example $u \neq 0$ and $v = u \times p$ gives $\omega(u, v) \neq 0$.
- The higher dimensional spheres S^{2n} with $n > 1$ don't carry a symplectic structure. This follows from the general fact, that the even dimensional De-Rham cohomology groups $H^k_{dR}(M)$ of a compact symplectic manifold M are not zero.
- Any non-orientable Manifold doesn't carry a symplectic structure, since ω^n is a non-vanishing form and therefore gives a orientation on any symplectic manifold.

Definition 1.6. Let $(M_1, \omega_1), (M_2, \omega_2)$ be two $2n$ dimensional symplectic manifolds. Let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a symplectomorphism if $\omega_1 = \varphi^*\omega_2$.

Theorem 1.3. *Let (m, ω) be a $2n$ dimensional symplectic manifold, and let $p \in M$ be any point in M. Then there is a coordinate chart $(U, \varphi = x_1, \dots, x_n, y_1, \dots, y_n)$ centered at p (i.e. $\varphi(p) = 0$) s.t.*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

locally on U. Such a chart is called a **Darboux chart**.

Example 1.2. The Cotangent Bundle
Let X be any n-dimensional manifold and let $M = T^*X$ be it's cotangent bundle. Let $(U, x_1, \ldots, x_n$ be a chart on X. Then the differentials $(dx_1)_x, \ldots, (dx_n)_x$ form a basis of T_x^*X for all $x \in U$. For $\xi \in T_x^*X$ we can write $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$. This induces a coordinate chart on $T^*U \subset T^*X$

$$\begin{array}{rcl} T^*U & \to & \mathbb{R}^{2n} \\ (x, \xi) & \mapsto & (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n). \end{array}$$

For two charts $(U, \varphi = x_1, \ldots, x_n), (V, \psi = x'_1, \ldots, x'_n)$ on X for $x \in U \cap V$ we have

$$(dx'_i)_x = \sum_{j=1}^n \frac{\partial(\psi \circ \varphi^{-1})_i}{\partial x_j}(\varphi(x))(dx_j)_x$$

and therefore for $\xi \in T_x^*X$ we have

$$\xi = \sum_{i=1}^n \xi_i (dx_i)_x = \sum_{j=1}^n \xi'_j (dx'_j)_x \tag{1}$$

where $\xi'_j = \sum_{i=1}^n \xi_i \dfrac{\partial(\psi \circ \varphi^{-1})_i}{\partial x_j}(\varphi(x))$. This is a smooth function and thus the cotangent bundle is smooth. We now have the following 2-form on T^*U,

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

This 2-form is closed since we have a 1-form on T^*U,

$$\alpha = \sum_{i=1}^n \xi_i dx_i$$

with $-d\alpha = \omega$. We claim that α is independent of the choice of coordinates. This follows directly from formula (1), since for 2 coordinate charts $(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, $(T^*V, x'_1, \ldots, x'_n, \xi'_1, \ldots, \xi'_n)$ we have that

$$\alpha = \sum_{i=1}^n \xi_i dx_i = \sum_{j=1}^n \xi'_j dx'_j = \alpha'$$

on $T^*U \cap T^*V$. So α gives a 1-form on T^*X called the ***tautological form*** or ***Liouville 1-form***, the 2-form ω is called the ***canonical symplectic form***.

Example 1.3. Complex projective Space $\mathbb{P}^n_{\mathbb{C}}$
Consider the linear coordinates $z_1, \ldots, z_n$ on $\mathbb{C}^n \setminus \{0\}$, where $z_j = x_j + iy_j$. We will use the complex valued forms

$$dz_j := dx_j + idy_j, \qquad d\overline{z}_j := dx_j - idy_j$$

and the differential operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

acting on the space of complex valued smooth functions on $\mathbb{C}^n \setminus \{0\}$. In this notation we can write the differential of a smooth function $f : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}$ as $df = \partial f + \overline{\partial} f$, where

$$\partial f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \qquad \overline{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.$$

We also need the 2-form

$$\partial\overline{\partial} f = \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k$$

We can now define a 2-form on $\mathbb{C}^n \setminus \{0\}$, by

$$\begin{aligned} \tilde{\omega}_{FS} &:= \frac{i}{2}\partial\overline{\partial}\log(|z|^2) \\ &= \frac{i}{2|z|^4} \sum_{j,k=1}^{n} \left(|z|^2 dz_k \wedge d\overline{z}_k - \overline{z}_j z_k dz_j \wedge d\overline{z}_k \right). \end{aligned}$$

Then there exists a unique real-valued 2-form ω_{FS} on $\mathbb{P}^{n-1}_{\mathbb{C}}$ s.t.

$$\tilde{\omega}_{FS} = \pi^* \omega_{FS}.$$

Where $\pi : \mathbb{C}^n \to \mathbb{P}^{n-1}_{\mathbb{C}}$ is the usual projection map. We call ω_{FS} the ***Fubini Study form*** on $\mathbb{P}^{n-1}_{\mathbb{C}}$. It will follow from the Marsden-Meyer-Weinstein Theorem that this is a symplectic form.

Definition 1.7. A ***submanifold*** of M is a manifold X which comes with a closed embedding $\iota : X \hookrightarrow M$, i.e. a proper injective immersion.

Definition 1.8. Let (M, ω) be a $2n$-dimensional symplectic manifold. A submanifold Y of M is called a ***Lagrangian submanifold*** if at each point $p \in Y$ the tangent space T_pY is a Lagrangian subspace of T_pM, i.e. $\omega_{p|T_pY} \equiv 0$ and $\dim(T_pY) = \frac{1}{2}\dim(T_pM)$.

Example 1.4. Let X be a manifold and $M = T^*X$ be the cotangent bundle. The zero section is the set

$$X_0 := \{(x, \xi) \in T^*X | \xi = 0 \text{ in } T^*_x X\}.$$

This is just the injection $X \hookrightarrow M$. We clearly have that $\alpha = \sum \xi_i dx_i$ restricts to 0 on X_0 and therefore $\omega_{|X_0} \equiv 0$.

Theorem 1.4. *Let $(M_1, \omega_1), (M_2, \omega_2)$ be two $2n$-dimensional symplectic manifolds. A diffeomorphism $\varphi : M_1 \to M_2$ is a symplectomorphism if and only if Γ_φ is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$, where $\tilde{\omega} = pr_1^*\omega_1 - pr_2^*\omega_2$.*

Proof. Let

$$\begin{aligned} \gamma : M_1 &\to M_1 \times M_2 \\ v &\mapsto (v, \varphi(v)) \end{aligned}$$

we have that Γ_φ is Lagrangian if and only if $\gamma^*\tilde{\omega} \equiv 0$. But we have that

$$\begin{aligned} \gamma^*\tilde{\omega} &= \gamma^* pr_1^*\omega_1 - \gamma^* pr_2^*\omega_2 \\ &= (pr_1 \circ \gamma)^*\omega_1 - (pr_2 \circ \gamma)^*\omega_2. \end{aligned}$$

But $pr_1 \circ \gamma$ is the identity map and $pr_1 \circ \gamma = \varphi$ and therefore

$$\gamma^* \tilde{\omega} \equiv 0 \Leftrightarrow \varphi^* \omega_2 = \omega_1.$$

□

Definition 1.9. Let $(M_1, \omega_1), (M_2, \omega_2)$ be two symplectic manifolds.A ***Lagrangian correspondence*** from $M_1 \to M_2$ is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.