Talk 1 & 2: Introduction to Toric Varieties

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1 Introduction

These are notes from the first two talks of the Variation of GIT seminar in the summer semester 2016, FU Berlin. In the two talks, we aim to give an introduction to toric variates and prove that normal semi-projective toric varieties can alternatively be constructed as a GIT quotient of affine space by a diagonalisable group linearised by a character. We will work with varieties over the complex numbers.

2 Toric Varieties

In this section we give an overview of constructions of toric varieties and describe the orbit-cone correspondence. The main reference for the results stated in this section are [2] §1.2 and [1] §1.2.

2.1 Definitions

Definition 2.1. A toric variety is a variety X which contains an algebraic torus ($T \cong (\mathbb{C}^*)^s$) as a dense open subset such that the action of the torus on itself extends to the whole of X.

Essentially toric varieties are just fattened tori with an action.

Example 2.2. The obvious example is a torus $X \cong (\mathbb{C}^*)^s$. Other easy examples are $\mathbb{C}^s$ and $\mathbb{P}^s$.

2.2 Affine Toric Varieties

1^{st} **Construction.**
We start with a lattice $N \cong \mathbb{Z}^s$ (that is, a free abelian group of finite rank). Then we define $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^*$ and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^s$. There is a natural pairing $\langle\, ,\, \rangle : M \times N \to \mathbb{Z}$. Let $X^*(T_N)$ (respectively $X_*(T_N)$) denote the group of characters (respectively the one parameter subgroups) of T_N. Then we have natural isomorphisms

$$\begin{aligned} M \cong X^*(T_N), \quad N \cong X_*(T_N) \\ m \mapsto \chi^m, \quad u \mapsto \lambda^u \end{aligned}$$

where $\chi^m\big(\sum_{i=1}^r u_i \otimes t_i \big) = \prod_{i=1}^r t_i^{\langle m, u_i \rangle}$, and $\lambda^u(t) = u \otimes t$. Now suppose we have a finite subset of M, $\mathcal{A} = \{m_1, ..., m_n\} \subset M$. When we see an element, $m \in M$ as a character of T, we denote it χ^m.

Now define

$$\Phi_{\mathcal{A}} : T_N \to \mathbb{C}^n,\ t \mapsto (\chi^{m_1}(t), ..., \chi^{m_n}(t)).$$

Define the toric variety associated to $\mathcal{A}$ as $X_{\mathcal{A}} := \overline{\mathrm{Im}(\Phi_{\mathcal{A}})}$.

Proposition 2.3. *For a finite subset $\mathcal{A} \subset M$, $X_{\mathcal{A}}$ is a toric variety.*

Proof. Define $T := \Phi_{\mathcal{A}}(T_N) \subseteq (\mathbb{C}^*)^n$. T is a torus (as an irreducible subvariety of a torus, which is also a subgroup, is again a torus), and is an open and dense subset of $X_{\mathcal{A}}$.

Now let us consider the action of T. Suppose $t \in T_N$, then $(\chi^{m_1}(t), ..., \chi^{m_n}(t)) \in T \subseteq (\mathbb{C}^*)^n$ acts by multiplication on $\mathbb{C}^n$ such that for every $Y \subset \mathbb{C}^n$, a closed subvariety, $t \cdot Y$ is again a closed subvariety. Then

$$T = t \cdot T \subseteq t \cdot X_{\mathcal{A}},$$

and taking closures we get that $X_{\mathcal{A}} \subseteq t \cdot X_{\mathcal{A}}$. Similarly, $X_{\mathcal{A}} \subseteq (t^{-1}) \cdot X_{\mathcal{A}}$, which implies $t \cdot X_{\mathcal{A}} \subseteq X_{\mathcal{A}}$ and thus $t \cdot X_{\mathcal{A}} = X_{\mathcal{A}}$. So the action is well defined on $X_{\mathcal{A}}$. ∎

Example 2.4. Suppose that $N = \mathbb{Z}^2$ and $\mathcal{A} = \{(0,1), (1,1), (2,1)\} \subset M$. Then $\Phi = \Phi_{\mathcal{A}} : T_N \to \mathbb{C}^2$ is given by $\Phi_{\mathcal{A}}(s,t) = (t, st, s^2 t)$. Note that

$$\mathrm{Im}(\Phi_{\mathcal{A}}) = \mathrm{Im}\big(v_2 : (a,b) \mapsto (a^2, ab, b^2)\big),$$

where v_2 is the second Veronese embedding. This is quick to see: Take $\Phi_{\mathcal{A}}(\frac{b}{a}, a^2) = (a^2, ab, b^2)$ and $v_2(t^{\frac{1}{2}}, t^{\frac{1}{2}} s) = (t, st, s^2 t)$. We recognise $\mathrm{Im}(\Phi)$ as the affine cone over the rational curve in $\mathbb{P}^2$, which we now know is an example of a toric variety.

2^{nd} **Construction** - Combinatorial construction.

We begin with some convex geometry. Again we start with a lattice N of rank s, and we denote its dual by M. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.5. A *rational convex polyhedral cone* in $N_{\mathbb{R}}$ is a subset of $N_{\mathbb{R}}$

$$\sigma = \mathrm{Cone}(S) = \{\Sigma_{u \in S} \lambda_u u \,|\, \lambda_u \geq 0\}$$

where $S \subset N$ is finite. We say that σ is generated by S.

Define

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \,|\, \langle m, u\rangle \geq 0, \ \forall u \in \sigma\}$$
$$\sigma^\perp = \{m \in M_{\mathbb{R}} \,|\, \langle m, u\rangle = 0, \ \forall u \in \sigma\}.$$

Remark 2.6. Note that $\sigma^\vee$ is a rational polyhedral cone in $M_{\mathbb{R}}$ (which is not necessarily convex) and that $(\sigma^\vee)^\vee = \sigma$.

Definition 2.7. Let $m \in M_{\mathbb{R}}$, define:

$$H_m := \{u \in N_{\mathbb{R}} \,|\, \langle m, u\rangle = 0\} \subset N_{\mathbb{R}},$$
$$H_m^+ := \{u \in N_{\mathbb{R}} \,|\, \langle m, u\rangle \geq 0\} \subset N_{\mathbb{R}}.$$

Remark 2.8. Note that for any convex polyhedral cone, σ, we have

$$\sigma = H_{m_1}^+ \cap ... \cap H_{m_s}^+$$

for some $m_1, ..., m_s \in M_{\mathbb{R}}$.

Definition 2.9.
1. A *face* of σ, is defined to be $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$. We write $\tau \leq \sigma$.
2. Define $\tau^* := \{m \in \sigma^\vee \,|\, \langle m, u\rangle = 0, \ \forall u \in \tau\} = \sigma^\vee \cap \tau^\perp$.
3. Define the *relative interior of* σ as $\mathrm{Relint}(\sigma) = \{u \in N_{\mathbb{R}} \,|\, \langle m, u\rangle > 0 \text{ for all } m \in \sigma^\vee \backslash \sigma^\perp\}$.

We now state some important properties of cones.

Lemma 2.10. *Let σ be a cone.*

1. *Every face of a cone is also a cone.*

2. *The intersection of two faces of σ is a face of σ.*

3. *A face of $\tau \leq \sigma$ is also a face of σ.*

Lemma 2.11. *Let σ_1 and σ_2 be polyhedral cones in $N_{\mathbb{R}}$ and let $\tau = \sigma_1 \cap \sigma_2$ be a common face. Then, for any $m \in \mathrm{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$,*

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2.$$

Definition 2.12 (Definition/Proposition)**.** We say that a cone $\sigma \subset N_{\mathbb{R}}$ is *strongly convex* if any of the following equivalent conditions hold:

1. $\{0\} \leq \sigma$,

2. $\sigma \cap (-\sigma) = \{0\}$,

3. $\dim \sigma^\vee = s$.

For the last condition, by dimension we mean $\dim \sigma = \dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}} \sigma$.

Lemma 2.13. *A strongly convex rational polyhedral cone is generated by the ray generators of its edges.*

Affine Semigroups and Affine Toric Varieties.

Definition 2.14. An affine semigroup is a semigroup whose associated $\mathbb{C}$-algebra is finitely generated. For a rational convex polyhedral cone σ, we define $S_\sigma = \sigma^\vee \cap M \subseteq M$.

We see now that S_σ is an affine semigroup.

Lemma 2.15 (Gordon's Lemma)**.** *Given a rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, then S_σ is a finitely generated semigroup.*

Corollary 2.16. *$\mathbb{C}[S_\sigma] = \langle \chi^m \,|\, m \in S_\sigma \rangle_{\mathbb{C}-alg}$ is a finitely generated $\mathbb{C}$-algebra.*

Definition 2.17. We define the affine toric variety associated to σ:

$$U_\sigma := \mathrm{Spec}\, \mathbb{C}[S_\sigma].$$

Now we show that U_σ is a toric variety. There are two ways of doing this. The first way is to pick a minimal finite generating set for S_σ, $\mathcal{A} = \{m_1, ..., m_n\} \subset M$. Then we claim that $X_{\mathcal{A}} \cong U_\sigma$, so that by the first construction U_σ is a toric variety. For a detailed proof see [1] Proposition 1.1.14. For the second method, we need a result about semigroup algebras ([2] §1.2 Proposition 2).

Lemma 2.18. *Suppose that $\tau \leq \sigma$ is a face of σ, i.e. $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$. Then we have the equality:*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-m).$$

Corollary 2.19. *With σ and τ as above, $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$, and thus U_τ is an open affine subset of U_σ. In particular for any strongly convex cone σ, as $\{0\}$ is a face of σ, we have that $U_{\{0\}} = T_N \subseteq U_\sigma$ for any σ.*

We define the torus action of T_N on U_σ as the following co-action:

$$\mathbb{C}[U_\sigma] \to \mathbb{C}[T_N] \otimes \mathbb{C}[U_\sigma], \quad \chi^m \mapsto \chi^m \otimes \chi^m.$$

Fans and Normal Toric Varieties

Definition 2.20. A *fan* $\Sigma \subseteq N_\mathbb{R}$ is a finite collection of strongly convex rational polyhedral cones $\sigma \subseteq N_\mathbb{R}$, such that

1. $\sigma \in \Sigma$ and $\tau \leq \sigma$ implies $\tau \in \Sigma$,
2. $\sigma_1, \sigma_2 \in \Sigma$ implies $\sigma_1 \cap \sigma_2 \leq \sigma_i$, for $i = 1, 2$.

We denote the set of r-dimensional cones of Σ by $\Sigma(r)$. The *support* of Σ is defined to be $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subseteq N_\mathbb{R}$.

We construct the toric variety X_Σ associated to Σ by gluing all the affine toric varieties U_σ associated to cones $\sigma \in \Sigma$ together as follows.

By Lemma 2.10, given any face τ of two cones σ_1, σ_2, such that $\tau = \sigma_1 \cap \sigma_2$, we know that

$$U_{\sigma_1} \supseteq U_\tau \subseteq U_{\sigma_2}.$$

Moreover, by Lemma 2.11, we have isomorphisms

$$g_{\sigma_1,\sigma_2} : \left(U_{\sigma_1}\right)_{\chi^m} \xrightarrow{\cong} \left(U_{\sigma_2}\right)_{\chi^{-m}},$$

which restricts to the identity on U_τ and together these morphisms give us enough data to glue. This doesn't immediately follow from results we saw in the talk, but a full proof can be found in [1]§ Theorem 3.1.5.

Theorem 2.21. *Suppose that $\Sigma \subseteq N_\mathbb{R}$ is a fan. Then X_Σ is a normal toric variety.*

Remark 2.22. In fact every separated normal toric variety comes from a fan, see [4].

2.3 Orbits and Cones

The aim of this section is to explain the bijective correspondence between T_N-orbits in X_Σ and cones σ in our fan, Σ. We start by introducing a very useful way to think of the closed points of a toric variety.

Proposition 2.23. *Let U_σ be an affine toric variety. There is a bijective correspondence between closed points of U_σ and semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$, where we consider $\mathbb{C}$ with multiplication as a semigroup.*

Proof. Suppose that $p \in V(\mathbb{C})$; then we define the semigroup homomorphism

$$\begin{aligned} \gamma_p : &S_\sigma \to \mathbb{C} \\ &m \mapsto \chi^m(p). \end{aligned}$$

For the other direction, given a semigroup homomorphism $\gamma : S_\sigma \to \mathbb{C}$. Then as $\{\chi^m\}_{m\in S_\sigma}$ form as basis of $\mathbb{C}[S_\sigma]$, we get a surjective homomorphism of $\mathbb{C}$-algebras.

$$\begin{aligned} \tilde{\gamma} : &\mathbb{C}[S_\sigma] \to \mathbb{C} \\ &\chi^m \mapsto \gamma(m). \end{aligned}$$

Then $\ker(\tilde{\gamma})$ is a maximal ideal of $\mathbb{C}[S_\sigma]$, which corresponds to a closed point in U_σ. It remains to see that these constructions are inverse to one another, see [1] Proposition 1.3.1 for the proof. ∎

We need the following technical result.

Proposition 2.24 ([1] Exercise 3.2.6)**.** *For a semigroup homomorphism $\gamma : S_\sigma \to \mathbb{C}$, there exists some face $\tau \leq \sigma$ such that*

$$\{m \in S_\sigma \,|\, \gamma(m) \neq 0\} = \tau^\perp \cap \sigma^\vee \cap M = \tau^\perp \cap S_\sigma.$$

Definition 2.25. For each cone $\sigma \in \Sigma$, we associate to σ a point γ_σ in U_σ, called the distinguished point, given by the semigroup homomorphism

$$\gamma_\sigma : S_\sigma \to \mathbb{C},\, m \mapsto \begin{cases} 1 & m \in S_\sigma \cap \sigma^\perp \\ 0 & \text{else.} \end{cases}$$

These distinguished points correspond to limits of one parameter subgroups, as described in the next proposition.

Proposition 2.26 ([1] 3.2.2.)**.** *Let σ be a rational polyhedral convex cone; then*

$$u \in \sigma \iff \lim_{t \to 0} \lambda^u(t) \text{ exists in } U_\sigma.$$

Moreover, if $u \in \mathrm{Relint}(\sigma)$, then $\lim_{t\to 0} \lambda^u(t) = \gamma_\sigma$

So now we have a way of representing cones as something geometric on the variety. So we study their associated orbits.

Definition 2.27. Suppose that Σ is a fan. Define the T_N-orbit in X_Σ associated to σ

$$O(\sigma) = T_N \cdot \gamma_\sigma.$$

Let us study the structure of the orbits. First we need a lemma.

Lemma 2.28. *Suppose that σ is a cone inside $N_\mathbb{R}$. Define $N_\sigma \subseteq N$ to be the sublattice spanned over $\mathbb{Z}$ by $\sigma \cap N$, and define*

$$N(\sigma) := \frac{N}{N_\sigma}.$$

Then

1. *The natural pairing $M \times N \to \mathbb{Z}$ induces a bilinear pairing*

$$\langle\, ,\, \rangle : (\sigma^\perp \cap M) \times N(\sigma) \to \mathbb{Z},$$

   which is perfect.

2. *The pairing above induces an isomorphism* $\mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}$.

Now we give an explicit description of the orbits.

Proposition 2.29 ([1] Lemma 3.2.5)**.** *Suppose that $\sigma \in \Sigma$ is a cone, then*

$$O(\sigma) = \left\{\gamma : S_\sigma \to \mathbb{C} \,|\, \gamma(m) \neq 0 \iff m \in (\sigma^\perp \cap M)\right\} \cong \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \cong T_{N(\sigma)}.$$

We are now ready to prove the theorem.

Theorem 2.30. *Let N be a lattice of rank n, let $\Sigma \subseteq N_\mathbb{R}$ be a fan and let X_Σ denote the associated toric variety. Then the following statements hold.*

1. *There is a bijective correspondence*

$$\{\text{cones in } \Sigma\} \longleftrightarrow \{T_N\text{- orbits in } X_\Sigma\}$$
$$\sigma \longmapsto O(\sigma).$$

2. $\dim O(\sigma) = n - \dim \sigma$.

3. $U_\sigma = \bigcup_{\tau \le \sigma} O(\tau)$.

4. $\tau \le \sigma \iff O(\sigma) \subseteq \overline{O(\tau)}$. *Hence*

$$\overline{O(\tau)} = \bigcup_{\tau \le \sigma} O(\sigma).$$

We give proofs of the first two parts; proofs for the others can be found in [1] Proposition 3.2.6.

Proof. 1. Suppose that O is a T_N-orbit in X_Σ. As the U_σ are T_N-invariant and cover X_σ, we have that $O \subseteq U_\sigma$, for some (actually many) $\sigma \in \Sigma$. As $U_\sigma \cap U_{\sigma'} = U_{\sigma \cap \sigma'}$, we can pick such a $\sigma \in \Sigma$ minimally, (that is, so that there are no faces $\tau \le \sigma$ such that $O \subseteq U_\tau$).

Now consider a point $\gamma \in O$. Regarding γ as a semigroup homomorphism, we know from Proposition 2.24 that there exists a face $\tau \le \sigma$ such that

$$\{m \in S_\sigma \,|\, \gamma(m) \neq 0\} = \tau^\perp \cap S_\sigma.$$

As τ is a face of σ, we know that there exists some $m' \in \sigma^\vee$, such that $\tau = H_{m'} \cap \sigma$, and that $S_\tau = S_\sigma + \mathbb{Z}_{\ge 0}(-m')$. Hence we can extend $\gamma : S_\sigma \to \mathbb{C}$ uniquely by sending $-m'$ to $\gamma(m')^{-1}$. This extension makes the following diagrams commute:

$$\begin{array}{ccc} S_\sigma & \xrightarrow{\gamma} & \mathbb{C} \\ {\scriptstyle i}\uparrow & \nearrow_{\gamma'} & \\ S_\tau & & \end{array} \qquad \begin{array}{ccc} \mathbb{C}[S_\sigma] & \xrightarrow{\tilde{\gamma}} & \mathbb{C} \\ {\scriptstyle i}\uparrow & \nearrow_{\tilde{\gamma'}} & \\ \mathbb{C}[S_\tau], & & \end{array}$$

where i is the inclusion map. Hence $\ker(\tilde{\gamma}) = \ker(\tilde{\gamma'})$, so they correspond to the same closed point. Therefore $\gamma \in U_\tau$ and hence $\tau = \sigma$, thus

$$\{m \in S_\sigma \,|\, \gamma(m) \neq 0\} = \sigma^\perp \cap M.$$

So by Proposition 2.29, $\gamma \in O(\sigma)$.

2. By Proposition 2.29 we have that $O(\sigma) = T_{N(\sigma)}$. Thus $\dim O(\sigma) = \dim N(\sigma)$. Consider the exact sequence:

$$0 \to N_\sigma \to N \to N(\sigma) \to 0.$$

Thus $\dim O(\sigma) = n - \dim \sigma$.

∎

2.4 Divisors on Toric Varieties

We begin with a classical result concerning Weil divisors on a normal variety.

Theorem 2.31. *Suppose that X is a normal variety and that $Z \subset X$ is a closed subvariety of codimension 1. Let $U = X \backslash Z$. There is a short exact sequence*

$$\mathbb{Z}^s \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0,$$

where s is the number of irreducible codimension 1 components of Z.

Proof. Suppose that Y is a prime divisor on X. Then $Y \cap U$ is either a prime divisor on U, or is empty. This gives us a map $\mu : \mathrm{Div}(X) \to \mathrm{Div}(U)$. Given a prime divisor on U, call it Y, then its closure in X is prime, and as $\mu(\overline{Y}) = Y$, μ is surjective. Now suppose that $Y = (f)$ is a principal divisor on X. Then $\mu(Y) = (f|_U)$ and hence μ descends to a surjective map $\mu : \mathrm{Cl}(X) \to \mathrm{Cl}(U)$.

Now define

$$\begin{aligned} \mathbb{Z}^s &\to \mathrm{Cl}(X) \\ (a_1, ..., a_s) &\mapsto \sum a_i Z_i. \end{aligned}$$

One can check that this generates the kernel of μ. ∎

Definition 2.32. Now suppose that $X = X_\Sigma$ is a normal toric variety associated to a fan $\Sigma \subseteq N_{\mathbb{R}}$. For each $\rho \in \Sigma(1)$ we can associated a prime divisor in the following way:

$$D_\rho := \overline{O(\rho)}.$$

Recall that the closure of an orbit is a union of orbits of smaller dimension, and so these are T_N-invariant. It is an easy exercise to check that any T_N-invariant prime divisor is in fact a D_ρ for some $\rho \in \Sigma(1)$.

We define the invariant group of divisors

$$\mathrm{Div}_{T_N}(X) = \{D = \sum a_i D_i \in \mathrm{Div}(X) \,|\, t \cdot D = \sum a_i (t \cdot D_i) = D, \, \forall t \in T_N\}.$$

One can show that this is freely generated by D_ρ, for $\rho \in \Sigma(1)$, [1] §4.1. Now suppose that $X = X_\Sigma$ is a normal toric variety associated to a fan $\Sigma \subseteq N_{\mathbb{R}}$. Then we know that T_N is an open dense subset of X, so let $Z := X \backslash T_N$, apply Theorem 2.31 and consider the the exact sequence

$$\mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to \mathrm{Cl}(T_N) \to 0.$$

Recall that $\mathrm{Cl}(T_N) = 0$ as $T_N = \mathrm{Spec}\, \mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$, and $\mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$ is a unique factorisation domain, see [3] II§.6. Hence we have a surjective homomorphism β,

$$\beta : \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X).$$

This tells us that in fact the T_N-invariant divisors generate the class group.

We define a group homomorphism $M \to \mathbb{Z}^{\Sigma(1)}$. Recall that $M \cong \{\chi^m : T_N \to \mathbb{C}^* \,|\, \chi^m$ a group homomorphism As each χ^m is also a morphism of affine varieties, we can consider its associated divisor, (χ^m). It is then clear that $\mathrm{Supp}(\chi^m) \subseteq \bigcup_\rho \mathrm{Supp}(D_\rho)$, as χ^m is well defined on T_N and the image is contained in $\mathbb{C}^*$. As $\chi^{m+m'} = \chi^m \cdot \chi^{m'}$, and hence $(\chi^{m+m'}) = (\chi^m) + (\chi^{m'})$, we get a group homomorphism

$$\begin{aligned} \alpha : M &\longrightarrow \mathrm{Div}_{T_N}(X), \\ m &\longmapsto (\chi^m). \end{aligned}$$

Note that $\mathrm{Div}_{T_N}(X) = \bigoplus_\rho \mathbb{Z} \cdot D_\rho$. In fact, given any rational function, f, such that $\mathrm{Supp}(f) \subseteq \bigcup_\rho \mathrm{Supp}(D_\rho)$, then we know that $f|_{T_N}$ will be a character, and hence $\ker(\alpha) = \mathrm{Im}(\beta)$. In fact, we have the following theorem.

Theorem 2.33. *Suppose that X_Σ is the toric variety associated to a fan $\Sigma \subseteq N_{\mathbb{R}}$, such that $\{u_\rho \,|\, \rho \in \Sigma(1)\}$ generates $N_{\mathbb{R}}$, where u_ρ is the minimal generator of ρ in N. Then there is a short exact sequence*

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho \to \mathrm{Cl}(X_\Sigma) \to 0.$$

Remark 2.34. For any normal toric variety X_Σ, we have proven that there is an exact sequence

$$M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X) \to 0.$$

In particular, we have shown that the class group is a finitely generated abelian group.

To prove the theorem, it remains to prove that α is injective. Before we can do this, we first need the following result.

Proposition 2.35 ([1] §4.1.1)**.** *For any toric variety $X = X_\Sigma$ associated to a fan Σ, we have*

$$\alpha(m) = \sum_\rho \langle m, u_\rho \rangle D_\rho.$$

Proof of Theorem 2.28. Given the extra hypothesis this is easy. Suppose that $m \in M$ is sent to zero, that is $\alpha(m) = 0$. By the previous proposition, this means that

$$\alpha(m) = \sum_\rho \langle m, u_\rho \rangle \cdot D_\rho = 0.$$

Hence, for each $\rho \in \Sigma(1)$, we have $\langle m, u_\rho \rangle = 0$, and, as the u_ρ span N, we can conclude $m = 0$, and α is injective, which completes the proof.

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2.5 Affine GIT

In progress.

2.6 Normal semi-projective toric varieties as GIT quotients

In progress.

References

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