Stratifications for group actions & moduli problems

Talk 5: Stratifications for moduli of sheaves

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§1 The Quot scheme & n-regularity

Let X be a projective variety $/\mathbb{C}$.
Let $\mathcal{O}(1)$ be an ample line bundle on X.
(i.e. a power of $\mathcal{O}(1)$ gives a proj. embedding $X \hookrightarrow \mathbb{P}^N$).

We consider coherent sheaves over X.

Def^n: For a coherent sheaf $\mathcal{E}$ over X, we define:

- $\mathcal{E}(n) := \mathcal{E} \otimes \mathcal{O}(1)^{\otimes n}$ Serre twist for $n \in \mathbb{Z}$
- $P(\mathcal{E}, n) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{E}(n))$ Hilbert polynomial of $\mathcal{E}$ w.r.t $\mathcal{O}(1)$.

We say $\mathcal{E}$ is n-regular if $H^i(X, \mathcal{E}(n-i)) = 0 \quad \forall\, i \geq 1$.

Facts about n-regularity (Castelnuovo-Mumford)

a) Any coherent sheaf $\mathcal{E}$ over X is n-regular for $n >> 0$.

b) If $\mathcal{E}$ is n-regular, then

(i) $\mathcal{E}$ is m-regular $\forall\, m \geq n$,

(ii) $H^i(\mathcal{E}(n)) = 0 \quad \forall\, i > 0$ (so $\dim H^0(\mathcal{E}(n)) = P(\mathcal{E}, n)$),

(iii) the evaluation map

$$\mathrm{eval}: H^0(\mathcal{E}(n)) \otimes \mathcal{O}(-n) \longrightarrow \mathcal{E} \quad \text{is surjective.}$$

(ie $\mathcal{E}(n)$ is generated by its global sections.)

The proof of both facts reduces to considering sheaves on $\mathbb{P}^r$ and then proceeds by induction on r.

Def^n: For a fixed coherent sheaf $\mathcal{F}$ over X & Hilbert poly. P let

$$\mathrm{Quot}_X(\mathcal{F}, P) := \left\{ q: \mathcal{F} \twoheadrightarrow \mathcal{E} \ \begin{array}{l} \text{surjection of coh} \\ \text{sheaves over } X \\ \text{s.t } P(\mathcal{E}) = P \end{array} \right\} \Big/ \sim$$

where $q \sim q' \iff \ker q = \ker q'$

can be highly singular.

This is the <u>Quot scheme</u> constructed by Grothendieck; it is a projective scheme of finite type over $\mathbb{C}$.

- $\mathrm{Quot}_X(\mathcal{F}, P)$ is an example of a fine moduli space: it represent the moduli problem of quotients of a fixed sheaf.
- $\mathrm{Quot}_X(\mathcal{F}, P)$ generalises the Grassmannian

$$\mathrm{Gr}(n,r) := \mathrm{Quot}_{\mathrm{Spec}\,\mathbb{C}}(\mathbb{C}^n, r) \quad \text{parametrises } r\text{-dim}^{e} \text{ quotients of } \mathbb{C}^n.$$

- In fact, the construction of Quot, uses an embedding into a Grassmannian

$$\mathrm{Quot}_X(\mathcal{F}, P) \hookrightarrow \mathrm{Gr}(H^0(\mathcal{F}(m)), P(m)) \quad \text{for } m \gg 0$$

$$(q: \mathcal{F} \twoheadrightarrow \mathcal{E}) \mapsto H^0(q(m)) : H^0(\mathcal{F}(m)) \twoheadrightarrow \underbrace{H^0(\mathcal{E}(m))}_{\dim = P(m)}$$

- The Hilbert schemes are special cases:

$$\mathrm{Hilb}_X(P) = \mathrm{Quot}_X(\mathcal{O}_X, P(\mathcal{O}_X) - P) = \{ \mathcal{O}_X \overset{q}{\twoheadrightarrow} \mathcal{F} \}/\sim$$
$$= \{ \mathcal{I}_Z = \ker q \hookrightarrow \mathcal{O}_X \}$$

<u>Lemma</u> Every n-regular sheaf on X with Hilbert poly P is parametrised by an open subscheme $Q^{n\text{-reg}} \subseteq \mathrm{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P)$ where

$$Q^{n\text{-reg}} := \left\{ q : \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) \twoheadrightarrow \mathcal{G} : \begin{array}{l} \mathcal{G} \text{ is } n\text{-regular} \\ \& \ H^0(q(n)) \text{ is an iso} \end{array} \right\} / \sim$$

Furthermore, for the natural $GL_{P(n)}$-action on $Q^{n\text{-reg}}$, we have

$$\{ GL_{P(n)}\text{-orbits on } Q^{n\text{-reg}} \} \xleftrightarrow{1:1} \left\{ \begin{array}{l} n\text{-regular sheaves over} \\ X \text{ w/ Hilbert poly } P \end{array} \right\} / \cong .$$

<u>Proof</u>: If $\mathcal{E}$ is n-regular then $H^i(\mathcal{E}(n)) = 0 \ \forall i > 0$ and $\mathrm{eval} : H^0(\mathcal{E}(n)) \otimes \mathcal{O}(-n) \to \mathcal{E}$ is surjective.

By choosing an isomorphism $H^0(\mathcal{E}(n)) \underset{\psi}{\cong} \mathbb{C}^{P(n)}$, we get a point $q_{\mathcal{E},\psi} \in Q^{n\text{-reg}}$.

$$\mathcal{E}' \cong \mathcal{E} \implies \begin{array}{ccc} H^0(\mathcal{E}(n)) & \cong & H^0(\mathcal{E}'(n)) \\ \psi \wr\wr & & \wr\wr \ \psi' \\ \mathbb{C}^{P(n)} & \xrightarrow{g} & \mathbb{C}^{P(n)} \end{array} \quad \& \quad g \cdot q_{\mathcal{E},\psi} = q_{\mathcal{E}',\psi'}$$

§2 Semistability & moduli of sheaves

There are two notions of semistability

1) Mumford's slope semistability for vector bundles
2) Gieseker-Maruyama reduced Hilbert polynomial semistability for pure sheaves ($\mathcal{F}$ s.t $\forall\, \mathcal{F}' \subseteq \mathcal{F}$ $\dim \mathrm{Supp}\, \mathcal{F}' = \dim \mathrm{Supp}\, \mathcal{F}$).

Over a curve, these notions coincide. In higher dimensions, they are different and, for the second, one can construct moduli spaces (following Gieseker, Maruyama & Simpson).

We'll use Rudakov's reformulation of 2):

Def^n A sheaf $\mathcal{E}$ over X is semistable (w.r.t $\mathcal{O}(1)$) if $\forall\, 0 \neq \mathcal{E}' \subseteq \mathcal{E}$, we have $\dfrac{P(\mathcal{E}', n)}{P(\mathcal{E}', m)} \leq \dfrac{P(\mathcal{E}, n)}{P(\mathcal{E}, m)}$ $\forall\, m >> n >> 0$

Notation $P(\mathcal{E}') \preccurlyeq P(\mathcal{E})$

Rmks
- With respect to $\preccurlyeq$, lower degree polynomials rank higher $\Rightarrow$ purity of $\mathcal{E}$ is necessary for semistability
- For polys of the same degree, $\preccurlyeq$ is equivalent to an inequality of reduced Hilbert polynomials.

Def^n: A collection of sheaves is bounded if they can be parametrised by a finite type scheme S.

$\Rightarrow \exists\, n$ such that every sheaf in this family is n-regular.

(If not, we'd have an ascending chain of closed subschemes of S which doesn't stabilise)

Theorem (Le Potier-Simpson)

The family of semistable sheaves over X with Hilbert polynomial P is bounded. Hence, for $n >> 0$ they are all n-regular.

For $n >> 0$, every semistable sheaf is parametrised by

$$Q^{n\text{-reg}}_{pure} = \left\{ q : \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) \twoheadrightarrow \mathcal{F} : \begin{array}{l} q \in Q^{n\text{-reg}} \\ \mathcal{F} \text{ is pure} \end{array} \right\} \subseteq \mathrm{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P)$$

Let R_n denote the closure of this subscheme.

For $m >> n$, consider Grothendieck's embedding $\mathrm{Quot} \hookrightarrow \mathrm{Grass}$ & let $L_{n,m} \to R_n$ be the ample line bundle for this embedding.

We can linearise the $SL_{P(n)}$-action on R_n to $L_{n,m} \to R_n$ and take the GIT quotient.

Theorem (Simpson)

$R_n /\!/_{L_{n,m}} SL_{P(n)}$ is a (coarse) moduli space for (S-equiv. classes of) semistable sheaves over X with Hilbert poly P.

§3 Harder-Narasimhan Stratifications (Shatz)

Defⁿ/Prop Every coherent sheaf $\mathcal{E}$ has a ! HN filtration $0 = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \cdots \subsetneq \mathcal{E}^{(s)} = \mathcal{E}$ such that $\mathcal{E}_i = \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ are semistable and $P(\mathcal{E}_1) \succ \cdots \succ P(\mathcal{E}_s)$

We define the HN type of $\mathcal{E}$ to be $\tau(\mathcal{E}) = (P(\mathcal{E}_1), \dots, P(\mathcal{E}_s))$

Idea: This combines the torsion filtration of $\mathcal{E}$ with the HN filtrations of the pure subquotients in the torsion filtration.

Theorem (Shatz)

Let $\mathcal{F} \to S \times X$ be a family of coherent sheaves over X with Hilbert polynomial P parametrised by S. Then there is a HN stratification of S: $S = \coprod_{\substack{\tau \\ \text{HN type}}} S_\tau$ where $S_\tau \subseteq S$ are locally closed.

Rmk: In general, there are infinitely many HN types

eg $X = \mathbb{P}^1$ $P(x) = 2(x+1)$ ie rk 2 degree zero v. bundles

For $n \in \mathbb{N}$, $\mathcal{E}_n = \mathcal{O}(n) \oplus \mathcal{O}(-n)$ has HN filtr $\mathcal{O}(n) \subseteq \mathcal{O}(n) \oplus \mathcal{O}(-n)$ and HN type $\tau_n = (x+n+1, x-n+1)$.

§4 A comparison of the stratifications on Quot

For $SL_{P(n)} \curvearrowright \mathrm{Quot}_n := \mathrm{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P)$ w.r.t. $L_{n,m}$ (and the Euclidean norm $\|-\|$ on $SL_{P(n)}$), we can consider the associated Hesselink stratification: $\mathrm{Quot}_n = \coprod_{\beta \in \mathcal{B}_{n,m}} S_\beta^{n,m}$

where $\beta = ([\lambda], d)$ or equivalently β is the conj class of a rational 1-PS λ_β.

We expect this stratification to agree with the stratification by HN types, following:

- the agreement result for quiver reps (see talk 4).
- Atiyah-Bott: The Yang-Mills stratification associated to a norm square of a moment map for a gauge gp $\mathcal{G}$ acting on an inf. dim^e space $\mathcal{A}$ of unitary connections on a fixed C^∞-v.bdle E over a curve C agrees with the HN stratification on the space $\mathcal{C} \cong \mathcal{A}$ of holo structures on E.

<u>Lemma</u>: The family of sheaves over X with HN type τ is bounded.

Hence, for $n \gg 0$, such sheaves are parametrised by $Q_\tau^{n\text{-reg}}$.

<u>Def^n</u>: For $\nu = (P_1, \dots, P_s)$ such that $\Sigma P_i = P$ and m, n, we define an associated Hesselink index $\beta_{n,m}(\nu) \in \mathcal{B}_{n,m}$ by the conj class of the rat^e 1-PS

$$t \mapsto \begin{pmatrix} t^{r_1} I_{P_1(n)} & & \\ & \ddots & \\ & & t^{r_s} I_{P_s(n)} \end{pmatrix}$$

where $r_i = \frac{P(m)}{P(n)} - \frac{P_i(m)}{P_i(n)}$.

<u>Rmk</u>:
- If ν is a HN type, then $P_1 > \dots > P_s$ and so for $m \gg n \gg 0$, $r_1 > \dots > r_s$.
- The rational weights r_i are picked to minimise the normalised Hilbert-Mumford weight.

<u>Theorem</u> (H.-Kirwan)

Let τ be a HN type; then for $m \gg n \gg 0$, we have

$$Q_\tau^{n\text{-reg}} \underset{\text{closed}}{\subseteq} S^{n,m}_{\beta_{n,m}(\tau)}$$

HN stratum; Hesselink stratum for $SL_{P(n)} \curvearrowright \mathrm{Quot}_n$ w.r.t. $L_{n,m}$

For the proof, we show an inclusion of the limit sets.

Question: Why don't these stratifications agree?

(1) For fixed n, m, the assignment $\text{HN types} \to \mathcal{B}_{n,m}$ (Hesselink indices), $\tau \longmapsto \beta_{n,m}(\tau)$ is not injective, unless X is a curve.

(2) This is an asymptotical statement for each HN type, but there are infinitely many HN types - can't pick n & m so the theorem holds for all HN types.
Moreover, the Hesselink stratification is finite.

(3) Quot schemes are only truncated parameter spaces: they do not parametrise all sheaves over X with Hilbert poly P.

$\rightsquigarrow$ Ideally want to compare the Hesselink stratifications of Quot_n for different n.

Rmk: The Hesselink strata are not connected and one can write $S_\beta^{n,m} = \coprod_\nu S_{\beta,\nu}^{n,m}$ where ν is a tuple of Hilbert polys that sum to P.

Then $Q_\tau^{n\text{-reg}} \underset{\text{closed}}{\subseteq} S_{\beta_{n,m}(\tau),\tau}$ for $m \gg n \gg 0$.

However, the presence of sheaves in Quot_n which are not n-regular prevents this from being an equality.

§5 An asymptotic Hesselink Stratification

For fixed n, for $m \gg n$, the Hesselink stratification of Quot_n w.r.t. $L_{n,m}$ stabilises. We write the refined stratification as

$$\text{Quot}_n = \coprod S_{\beta,\nu}^n .$$

We want to compare these stratifications as n increases, but there is no natural maps $\text{Quot}_n \longrightarrow \text{Quot}_{n'}$ for $n' > n$.
However, as every n-regular sheaf is n'-regular for $n' > n$, we have morphisms

$$\begin{array}{ccc} Q^{n\text{-reg}} & \longrightarrow & Q^{n'\text{-reg}} \\ \circlearrowleft & & \circlearrowleft \\ GL_{P(n)} & \hookrightarrow & GL_{P(n')} \end{array}$$

which are equivariant.

In fact, if we take the stack quotient, we get

$$(*)\quad \mathrm{Coh}^{n\text{-reg}}_{X,P} = \left[Q^{n\text{-reg}} / GL_{P(n)} \right] \hookrightarrow \mathrm{Coh}^{n'\text{-reg}}_{X,P} = \left[Q^{n'\text{-reg}} / GL_{P(n')} \right]$$

stack of n-regular coherent sheaves on X w/ H. poly P.

For each n, the Hesselink stratification of Quot_n can be restricted to $Q^{n\text{-reg}}$ and, as the strata are $GL_{P(n)}$-invariant, we get an induced stratification $\mathrm{Coh}^{n\text{-reg}}_{X,P} = \coprod_{\beta,\nu} \mathcal{S}^n_{\beta,\nu}$.

The limit of $(*)$ is the <u>stack</u>

$\mathrm{Coh}_{X,P}$ of coherent sheaves over X with Hilbert poly. P.

This is the correct space to construct an asymptotic Hesselink stratification.

<u>Notation</u>: $\mathcal{S}^n_\nu = \mathcal{S}^n_{\beta_n(\nu),\nu}$ for $\nu = (P_1, \dots, P_s)$ s.t. $\sum P_i = P$.

<u>Theorem</u> (H.)

For $n' > n$, let

$$\begin{tikzcd} \mathcal{S}^{n,n'}_\nu \arrow[r, hook] \arrow[d, hook] & \mathcal{S}^{n'}_\nu \arrow[d, hook] \\ \mathcal{S}^n_\nu \arrow[r, hook] & \mathrm{Coh}^{n'\text{-reg}}_{X,P} \end{tikzcd}$$

be the fibre product.

Then for $n' \gg n \gg 0$, the stacks $\mathcal{S}^{n,n'}_\nu$ stabilise to $\mathcal{S}_\nu$ where $\mathcal{F} \in \mathcal{S}_\nu \iff \mathcal{F} \in \mathcal{S}^n_\nu \;\forall n \gg 0$.

Furthermore $\mathcal{S}_\nu \neq \emptyset \iff \nu$ is a HN type.

The proof relies on the previous theorem.

<u>Theorem</u> (H.)

Let $(X, \mathcal{O}(1))$ be a polarised proj. variety & P be a Hilbert poly.

On $\mathrm{Coh}_{X,P}$, the following stratifications agree:

(1) the asymptotic Hesselink stratification $\mathrm{Coh}_{X,P} = \coprod_\nu \mathcal{S}_\nu$,

(2) the stratification by HN types $\mathrm{Coh}_{X,P} = \coprod_\tau \mathrm{Coh}^\tau_{X,P}$.